ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to Robert Frontczak, LBBW, Am Hauptbahnhof 2, 70173 Stuttgart, Germany, or by e-mail at the address robert.frontczak@lbbw.de. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-906 Proposed by Kenny B. Davenport, Dallas, PA

Prove the identity

$$3\left(\sum_{k=1}^{n} F_{k-3}F_{k-2}F_{k-1}F_{k+1}F_{k+2}F_{k+3} - \sum_{k=1}^{n} F_k^6\right) = 4(-1)^n(F_nF_{n+1})^2 - 11F_nF_{n+1} + 12D_n,$$

where $D_n = (1+(-1)^n)/2$ is 0 if n is odd and 1 if n is even.

H-907 Proposed by Andrés Ventas, Santiago de Compostela, Spain, and Curtis Cooper, Warrensburg, MO

Let $n \geq 0$ be an integer. Prove that

$$F_{3n} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-i-1}{i} 2^{2n-1-4i}.$$

H-908 Proposed by D. M. Bătineţu Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania

Prove that in every triangle ABC with area F and altitudes h_a, h_b, h_c perpendicular to the sides a, b, c, respectively, the following inequalities hold:

(i) $$\frac{a^{F_n}b^{F_{n+2}}}{h_a^{F_{n+1}}} + \frac{b^{F_n}c^{F_{n+2}}}{h_b^{F_{n+1}}} + \frac{c^{F_n}a^{F_{n+2}}}{h_c^{F_{n+1}}} \geq 2^{F_n+F_{n+2}}\sqrt{3}^{2-F_{n+2}}F^{F_n};$$

(ii) $$\frac{a^{F_n^2}b^{F_{2n+1}}}{h_a^{F_{n+1}^2}} + \frac{b^{F_n^2}c^{F_{2n+1}}}{h_b^{F_{n+1}^2}} + \frac{c^{F_n^2}a^{F_{2n+1}}}{h_c^{F_{n+1}^2}} \geq 2^{F_{2n+1}+F_n^2}\sqrt{3}^{2-F_{2n+1}}F^{F_n^2}.$$

H-909 Proposed by Michel Bataille, Rouen, France

Let n be a positive integer. For each integer k in $[0, n/2]$, let

$$U_{n,k} = \sum_{j=0}^{k}\binom{k}{j}\frac{4^j}{n-j} \qquad \text{and} \qquad V_{n,k} = \sum_{j=0}^{k}\binom{n}{j}(-5)^j.$$

Prove that

$$n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}U_{n,k}=2^nL_n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}\frac{(-1)^kV_{n,k}}{\binom{n-1}{k}}.$$

H-910 Proposed by Robert Frontczak, Stuttgart, Germany

Show the following identity involving Lucas numbers holds:

$$\sum_{k=1}^{\infty}\frac{L_{k+1}}{k(k+1)^2 2^{k+1}}=1-\frac{\pi^2}{12}+\frac{\ln(2)}{2}(2\ln(2)-3)-\ln(\alpha)(2\ln(\alpha)-\sqrt{5}).$$

SOLUTIONS

Fibonacci numbers and the alternating Riemann zeta function

H-872 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 59, No. 1, February 2021)

Prove that

$$\sum_{n=1}^{\infty}\eta(2n)\frac{F_{2n}}{5^n}=\frac{\pi}{10\cos(\frac{\pi}{2\sqrt{5}})}\quad\text{and}\quad\sum_{n=1}^{\infty}\eta(2n)\frac{L_{2n}}{5^n}=\frac{\pi}{2\cos(\frac{\pi}{2\sqrt{5}})}-1,$$

where $\eta(s)=\sum_{k=1}^{\infty}(-1)^{k-1}/k^s$ (defined for $\mathrm{Re}(s)>0$) is the Dirichlet η (or alternating Riemann zeta) function.

Solution by the proposer

We will need the following lemma.

Lemma. *For all complex numbers z with $|z|<\pi$ it holds that*

$$\sum_{n=1}^{\infty}\eta(2n)z^{2n}=\frac{1}{2}\left(\frac{\pi z}{\sin(\pi z)}-1\right).$$

Proof. The connection between the Riemann zeta function and its alternating variant is

$$\eta(s)=(1-2^{1-s})\zeta(s).$$

Therefore, using

$$\sum_{n=1}^{\infty}\zeta(2n)z^{2n}=\frac{1}{2}\left(1-\pi z\cot(\pi z)\right),\qquad(|z|<\pi),$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty}\eta(2n)z^{2n} &= \sum_{n=1}^{\infty}\zeta(2n)z^{2n}-2\sum_{n=1}^{\infty}\zeta(2n)\left(\frac{z}{2}\right)^{2n}\\ &= -\frac{1}{2}-\frac{\pi z}{2}\left(\cot(\pi z)-\cot\left(\frac{\pi z}{2}\right)\right)\\ &= -\frac{1}{2}-\frac{\pi z}{2}\left(\frac{\cos(\pi z)}{\sin(\pi z)}-\frac{\cos(\pi z/2)}{\sin(\pi z/2)}\right),\end{aligned}$$

and the statement follows upon using the addition formula for the sine function $\sin(x+y)=\sin x\cos y+\cos x\sin y$. □

Having the above result in hand, we can prove the first identity directly calculating

$$\sum_{n=1}^{\infty}\eta(2n)\frac{F_{2n}}{5^n} = \frac{1}{\sqrt{5}}\left(\sum_{n=1}^{\infty}\eta(2n)\left(\frac{\alpha}{\sqrt{5}}\right)^{2n} - \sum_{n=1}^{\infty}\eta(2n)\left(\frac{\beta}{\sqrt{5}}\right)^{2n}\right)$$
$$= \frac{1}{2\sqrt{5}}\left(\frac{\pi\alpha/\sqrt{5}}{\sin(\pi\alpha/\sqrt{5})} - \frac{\pi\beta/\sqrt{5}}{\sin(\pi\beta/\sqrt{5})}\right).$$

Because

$$\frac{\pi\alpha}{\sqrt{5}} = \frac{\pi}{2} + \frac{\pi}{2\sqrt{5}} \qquad \text{and} \qquad \frac{\pi\beta}{2\sqrt{5}} = \frac{\pi}{2} - \frac{\pi}{2\sqrt{5}},$$

the result follows by inserting the above formulas and simplifying using the addition formula for the sine function. The second identity is proved in the same manner, and the proof is omitted.

Editor's note: Andrés Ventas points out that a generalized version of this problem has appeared in [1].

Reference

[1] K. Boyadzhiev and R. Frontczak, *Series involving Euler's Eta (or Dirichlet's Eta) Function*, J. Integer Sequences, **24** (2021), Article 21.9.1.

Also solved by Brian Bradie, Dmitry Fleischman, Haydn Gwyn, Raphael Schumacher, Albert Stadler, Séan M. Stewart, and David Terr.

Identities with Fibonacci and Tribonacci numbers

H-873 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 59, No. 2, May 2021)

Let $(T_n)_{n\geq 0}$ be the Tribonacci sequence defined by $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geq 0$ with $T_0 = 0$, $T_1 = T_2 = 1$. Prove the following identities valid for all $n \geq 2$:

(i)

$$T_n = (-1)^{n+1}F_n + 2(-1)^n F_{n-1} + \sum_{k=0}^{n-2}(-1)^{k+1}F_k(2T_{n-k} + T_{n-2-k}).$$

(ii)

$$\sum_{1\leq i<j\leq n}(F_j - F_i)(T_{n-j} - T_{n-i}) = n(T_{n+2} - F_{n+2}) - \frac{1}{2}(F_{n+2} - 1)(T_{n+1} + T_{n-1} - 1).$$

(iii)

$$\sum_{1\leq i<j\leq n}(L_j - L_i)(T_{n-j} - T_{n-i}) = n(2T_{n+3} - T_{n+2} - 2T_n - L_{n+2}) - \frac{1}{2}(L_{n+2} - 3)(T_{n+1} + T_{n-1} - 1).$$

Solution by Andrés Ventas, Santiago de Compostela, Spain

For (i) we use the same method as used in [1]. We also need the generating function for the alternating Fibonacci sequence obtained with Maxima (see [2]):

$$f_a(x) = ggf([0, 1, -1, 2, -3, 5, -8, 13, -21, 34, -55]) = -\frac{x}{x^2 - x - 1}.$$

$$f_a(x) = \frac{x}{1+x-x^2}. \quad u(x) = \frac{x}{1-x-x^2-x^3}.$$
$$1+x-x^2-2x-x^3 = 1-x-x^2-x^3 \Rightarrow \frac{x}{f_a(x)} - 2x - x^3 = \frac{x}{u(x)}.$$
$$u(x) = f_a(x) + f_a(x)u(x)(2+x^2).$$
$$T_n = (-1)^{n+1}F_n + \sum_{k=0}^{n}(-1)^{k+1}F_kT_{n-k}\cdot 2 + \sum_{k=0}^{n-2}(-1)^{k+1}F_kT_{n-2-k}.$$
$$T_n = (-1)^{n+1}F_n + 2(-1)^nF_{n-1} + \sum_{k=0}^{n-2}(-1)^{k+1}F_k(2T_{n-k}+T_{n-2-k}).$$

For (ii) and (iii) we use the Binet–Cauchy identity (see [3]):

$$\left(\sum_{i=1}^{n}a_ic_i\right)\left(\sum_{j=1}^{n}b_jd_j\right) = \left(\sum_{i=1}^{n}a_id_i\right)\left(\sum_{j=1}^{n}b_jc_j\right) + \sum_{1\le i<j\le n}(a_ib_j-a_jb_i)(c_id_j-c_jd_i),$$

and several identities obtained or cited in Theorems 2.1, 2.3, Identities 3.7, 3.10, 3.30:

$$\text{(Theorem 2.1) } T_n = F_n + \sum_{k=0}^{n-2}F_kT_{n-2-k} \Rightarrow \sum_{k=1}^{n}F_kT_{n-k} = T_{n+2} - F_{n+2} - F_0T_n.$$
$$\text{(Theorem 2.3) } 2T_n = T_{n-1} + L_{n-1} + \sum_{k=0}^{n-3}L_kT_{n-3-k} \Rightarrow$$
$$\sum_{k=1}^{n}L_kT_{n-k} = 2T_{n+3} - T_{n+2} - L_{n+2} - L_0T_n.$$
$$\text{(Id. 3.7) } \sum_{n=1}^{N}F_n = F_{N+2} - 1.$$
$$\text{(Id. 3.10) } \sum_{n=1}^{N}T_n = \frac{1}{2}(T_{N+2}+T_N-1). \qquad \text{(Id. 3.30) } \sum_{n=1}^{N}L_n = L_{N+2} - 3.$$

(ii) Substituting $a_i = 1$, $d_i = 1$, $b_i = F_i$, $b_j = F_j$, $c_i = T_{n-i}$, and $c_j = T_{n-j}$ into the Binet–Cauchy identity we get

$$\left(\sum_{i=1}^{n}T_{n-i}\right)\left(\sum_{j=1}^{n}F_j\right) = \left(\sum_{i=1}^{n}1\right)\left(\sum_{j=1}^{n}F_jT_{n-j}\right) + \sum_{1\le i<j\le n}(F_j-F_i)(T_{n-i}-T_{n-j}) \Rightarrow$$
$$\sum_{1\le i<j\le n}(F_j-F_i)(T_{n-j}-T_{n-i}) = \left(\sum_{i=1}^{n}1\right)\left(\sum_{j=1}^{n}F_jT_{n-j}\right) - \left(\sum_{i=1}^{n}T_{n-i}\right)\left(\sum_{j=1}^{n}F_j\right).$$
$$\sum_{1\le i<j\le n}(F_j-F_i)(T_{n-j}-T_{n-i}) = n(T_{n+2}-F_{n+2}) - \frac{1}{2}(T_{n+1}+T_{n-1}-1)(F_{n+2}-1).$$

And now we do the same for (iii), substituting $a_i = 1$, $d_i = 1$, $b_i = L_i$, $b_j = L_j$, $c_i = T_{n-j}$, and $c_j = T_{n-i}$ into the Binet–Cauchy identity and using the identities for Lucas numbers.

$$\left(\sum_{i=1}^{n} T_{n-i}\right)\left(\sum_{j=1}^{n} L_j\right) = \left(\sum_{i=1}^{n} 1\right)\left(\sum_{j=1}^{n} L_j T_{n-j}\right) + \sum_{1\le i<j\le n} (L_j - L_i)(T_{n-i} - T_{n-j}) \Rightarrow$$

$$\sum_{1\le i<j\le n} (L_j - L_i)(T_{n-j} - T_{n-i}) = \left(\sum_{i=1}^{n} 1\right)\left(\sum_{j=1}^{n} L_j T_{n-j}\right) - \left(\sum_{i=1}^{n} T_{n-i}\right)\left(\sum_{j=1}^{n} L_j\right).$$

$$\sum_{1\le i<j\le n} (L_j - L_i)(T_{n-j} - T_{n-i}) = n(2T_{n+3} - T_{n+2} - L_{n+2} - 2T_n)$$

$$-\frac{1}{2}(T_{n+1} + T_{n-1} - 1)(L_{n+2} - 3).$$

References

[1] R. Frontczak, *Some Fibonacci-Lucas-Tribonacci-Lucas identities*, The Fibonacci Quarterly, **56.3** (2018), 263–274.
[2] Maxima.sourceforge.io, Maxima, a Computer Algebra System, Version 5.45.1 (2021) `https://maxima.sourceforge.io/`
[3] Wikipedia, Binet-Cauchy identity, `https://en.wikipedia.org/wiki/Binet-Cauchy_identity/`

Also solved by Dmitry Fleischman, Albert Stadler, and the proposer.

Catalan meets Fibonacci

H-874 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 59, No. 2, May 2021)

Let C_n be the nth Catalan number; i.e., $C_n = \frac{1}{n+1}\binom{2n}{n}$, and α be the golden section. Prove that

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{n(n+1)C_n} = \alpha^{-2} \sum_{n=1}^{\infty} \frac{L_{2n}}{n(n+1)C_n} = 2\pi\sqrt{\frac{\alpha}{25\sqrt{5}}}.$$

Solution by Michel Bataille, Rouen, France

We use the following theorem, stated and proved in [1]: If $|x| < 1$, then

$$\sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m\binom{2m}{m}} = \frac{2x\arcsin(x)}{\sqrt{1-x^2}}.$$

Because $\left|\frac{\alpha}{2}\right|, \left|\frac{\beta}{2}\right| < 1$, this theorem yields

$$\sum_{n=1}^{\infty} \frac{\alpha^{2n}}{n\binom{2n}{n}} = \frac{2\alpha\arcsin(\alpha/2)}{\sqrt{4-\alpha^2}}, \qquad \sum_{n=1}^{\infty} \frac{\beta^{2n}}{n\binom{2n}{n}} = \frac{2\beta\arcsin(\beta/2)}{\sqrt{4-\beta^2}}.$$

We have $\frac{\alpha}{2} = \frac{\sqrt{5}+1}{4} = \cos\frac{\pi}{5}$; hence, $\arcsin(\alpha/2) = \frac{\pi}{2} - \arccos(\alpha/2) = \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$ and $4 - \alpha^2 = 3 - \alpha$. Similarly, we find $\arcsin(\beta/2) = -\frac{\pi}{10}$ and $4 - \beta^2 = 3 - \beta$. If $S = \sum_{n=1}^{\infty} \frac{F_{2n}}{n(n+1)C_n}$, it follows that

$$S = \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}\left(\frac{\alpha^{2n}}{n\binom{2n}{n}} - \frac{\beta^{2n}}{n\binom{2n}{n}}\right) = \frac{1}{\sqrt{5}}\cdot\frac{\pi}{10}\cdot 2\left(\frac{3\alpha}{\sqrt{3-\alpha}} + \frac{\beta}{\sqrt{3-\beta}}\right)$$
$$= \frac{\pi}{5\sqrt{5}}\cdot\frac{3\alpha\sqrt{3-\beta}+\beta\sqrt{3-\alpha}}{\sqrt{(3-\alpha)(3-\beta)}}.$$

Easy calculations give $\sqrt{(3-\alpha)(3-\beta)} = \sqrt{5}$ and $(3\alpha\sqrt{3-\beta}+\beta\sqrt{3-\alpha})^2 = 50+10\sqrt{5}$ so that

$$S = \frac{\pi}{5\sqrt{5}}\cdot\frac{1}{\sqrt{5}}\sqrt{50+10\sqrt{5}} = \frac{\pi\sqrt{10}}{25}\sqrt{5+\sqrt{5}} = \frac{\pi\sqrt{10}}{25}\sqrt{\frac{10\alpha}{\sqrt{5}}} = 2\pi\sqrt{\frac{\alpha}{25\sqrt{5}}}.$$

In the same way, we calculate $T = \sum\limits_{n=1}^{\infty}\frac{L_{2n}}{n(n+1)C_n}$ as follows:

$$T = \sum_{n=1}^{\infty}\left(\frac{\alpha^{2n}}{n\binom{2n}{n}} + \frac{\beta^{2n}}{n\binom{2n}{n}}\right) = \frac{\pi}{5}\left(\frac{3\alpha}{\sqrt{3-\alpha}} - \frac{\beta}{\sqrt{3-\beta}}\right) = \frac{\pi}{5\sqrt{5}}\sqrt{50+22\sqrt{5}}.$$

The required result follows because $2\sqrt{5}\alpha^2\sqrt{\frac{\alpha}{\sqrt{5}}} = (5+3\sqrt{5})\sqrt{\frac{5+\sqrt{5}}{10}} = \sqrt{50+22\sqrt{5}}$ is readily checked.

Reference

[1] D. H. Lehmer, *Interesting series involving the central binomial coefficient*, Amer. Math. Monthly, **92.7** (1985), 452.

Also solved by Brian Bradie, Dmitry Fleischman, Hideyuki Ohtsuka, Raphael Schumacher, Albert Stadler, Séan M. Stewart, Andrés Ventas, and the proposer.

A geometric inequality involving Fibonacci numbers

H-875 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania
(Vol. 59, No. 2, May 2021)

Let ABC be a triangle with a, b, c the lengths of the sides, R the length of the circumradius, r the length of the inradius, and s the semiperimeter. Prove that

$$\left(\frac{F_n^2a^2+F_{n+1}^2b^2}{c}\right)^2 + \left(\frac{F_n^2b^2+F_{n+1}^2c^2}{a}\right)^2 + \left(\frac{F_n^2c^2+F_{n+1}^2a^2}{b}\right)^2 \geq 2F_{2n+1}^2(s^2-r^2-4Rr)$$

holds for all $n \geq 0$.

Solution by Brian Bradie, Newport News, VA

By the Cauchy-Schwarz inequality,

$$\left(\frac{F_n^2a^2+F_{n+1}^2b^2}{c}\right)^2 + \left(\frac{F_n^2b^2+F_{n+1}^2c^2}{a}\right)^2 + \left(\frac{F_n^2c^2+F_{n+1}^2a^2}{b}\right)^2$$
$$\geq \frac{((F_n^2+F_{n+1}^2)(a^2+b^2+c^2))^2}{a^2+b^2+c^2}$$
$$= F_{2n+1}^2(a^2+b^2+c^2).$$

The desired result follows from the identity

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr).$$

Also solved by Michel Bataille, Dmitry Fleischman, Wei-Kai Lai, Nandan Sai Dasireddy, Albert Stadler, Andrés Ventas, and the proposers.

Inequalities with Fibonacci and Lucas numbers

H-876 Proposed by I. V. Fedak, Ivano-Frankivsk, Ukraine (Vol. 59, No. 2, May 2021)

For all positive integers n, prove that

$$F_{n+2} \geq \sqrt{\frac{F_nF_{n+1}+1}{n+1}} + n\sqrt[n+1]{F_1F_2\cdots F_n}; \qquad L_{n+2} \geq \sqrt{\frac{L_nL_{n+1}+1}{n+3}} + (n+2)\sqrt[n+3]{L_1L_2\cdots L_n}.$$

Solution by Michel Bataille, Rouen, France

We first show the following lemma.

Lemma. *Let a_1, a_2, ..., a_n be positive real numbers with sum S, geometric mean G, and quadratic mean Q. Then, $S \geq Q + (n-1)G$.*

Proof. Because $a_1^2 + a_2^2 + \cdots + a_n^2 = nQ^2$ and

$$\sum_{1\leq i<j\leq n} a_ia_j \geq \frac{n(n-1)}{2}(a_1^{n-1}\cdot a_2^{n-1}\cdots a_n^{n-1})^{\frac{2}{n(n-1)}} = \frac{n(n-1)}{2}G^2$$

(using the arithmetic mean-geometric mean inequality), we obtain $S^2 \geq nQ^2 + n(n-1)G^2$. Therefore, it is sufficient to prove that

$$nQ^2 + n(n-1)G^2 \geq Q^2 + (n-1)^2G^2 + 2(n-1)QG.$$

We are done because this becomes $(n-1)(Q-G)^2 \geq 0$, which obviously holds. □

We first apply the lemma with $n+1$ instead of n and with $a_1 = F_1$ and $a_k = F_{k-1}$ for $k = 2, 3, \ldots, n+1$. We obtain

$$F_1 + \sum_{k=1}^{n} F_k \geq \sqrt{\frac{F_1^2 + \sum_{k=1}^n F_k^2}{n+1}} + n\sqrt[n+1]{F_1\cdot F_1\cdot F_2\cdots F_n}. \tag{1}$$

Because $F_1 = 1$, $\sum_{k=1}^n F_k = F_{n+2} - 1$ and $\sum_{k=1}^n F_k^2 = F_nF_{n+1}$, (1) rewrites as

$$F_{n+2} \geq \sqrt{\frac{F_nF_{n+1}+1}{n+1}} + n\sqrt[n+1]{F_1F_2\cdots F_n}.$$

Second, we apply the lemma with $n+3$ instead of n and with $a_1 = a_2 = a_3 = L_1 = 1$ and $a_k = L_{k-3}$ for $k = 4, 5, \ldots, n+3$. Similarly, because $\sum_{k=1}^n L_k = L_{n+2} - 3$ and $\sum_{k=1}^n L_k^2 = L_nL_{n+1} - 2$, we obtain

$$L_{n+2} \geq \sqrt{\frac{L_nL_{n+1}+1}{n+3}} + (n+2)\sqrt[n+3]{L_1L_2\cdots L_n}.$$

Also solved by Ilia Antypenko, Brian Bradie, Dmitry Fleischman, Albert Stadler, Andrés Ventas, and the proposer.

Some formulas involving powers of Lucas numbers

H-877 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 59, No. 2, May 2021)

Given an even integer r and an integer $n \geq 0$, prove that

$$\sum_{k=0}^{n} \binom{2n-k}{n} L_r^k L_{r(k+1)} = L_r^{2n+1}.$$

Solution by Robert Frontczak, Stuttgart, Germany

The proposed identity is not new and appeared in [1]. In [1], equation (3.2), a generalization of the proposal is stated in the equivalent form

$$\sum_{k=1}^{n} \binom{2n-1-k}{n-1} L_r^k L_{rk} = (-1)^{rn} L_r^{2n},$$

which reduces to the proposed identity when r is even. The identities reappeared recently in [2].

References

[1] P. Filipponi, *Some binomial Fibonacci identities*, The Fibonacci Quarterly, **33.3** (1995), 251–257.

[2] R. Frontczak and T. Goy, *Combinatorial sums associated with balancing and Lucas balancing polynomials*, Annales Math. et Inf., **52** (2020), 97–105.

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Raphael Schumacher, Albert Stadler, and the proposer.

Acknowledgement. Albert Stadler solved **H-871**.