ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-787 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{2F_{n+1}} F_{2F_n} F_{2F_{n+2}}} = \frac{7-3\sqrt{5}}{2}.$$

H-788 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Given $c > 0$ determine

$$\lim_{n\to\infty} \sqrt{cF_2^2 + \sqrt{cF_4^2 + \sqrt{cF_8^2 + \sqrt{\cdots + \sqrt{cF_{2^n}^2}}}}}.$$

H-789 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

For any real numbers $x,\ y$ we denote $B(x,y) = \sqrt{\frac{x^2+xy+y^2}{3}}$. Prove that for $n \ge 1$, we have

(i) $$\left(\frac{L_{n+2}-3}{n}\right)^2 \le \frac{1}{n} \sum_{n \text{ cyclic}} B^2(L) \le \frac{L_nL_{n+1}-2}{n};$$

(ii) $$\left(\frac{F_{n+2}-1}{n}\right)^2 \le \frac{1}{n} \sum_{n \text{ cyclic}} B^2(F) \le \frac{F_nF_{n+1}}{n},$$

where for a sequence $X := \{X_m\}_{m\ge 1}$ we use

$$\sum_{n \text{ cyclic}} B^2(X) = B^2(X_1,X_2) + B^2(X_2,X_3) + \cdots + B^2(X_{n-1},X_n) + B^2(X_n,X_1).$$

H-790 **Proposed by Ovidiu Furdui, Cluj-Napoca, Romania.**

Calculate

$$\sum_{n=2}^{\infty}\left(H_n-\gamma-\frac{\zeta(2)}{2}-\frac{\zeta(3)}{3}-\cdots-\frac{\zeta(n)}{n}\right),$$

where ζ denotes the Riemann zeta function and $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ is the nth harmonic number.

H-791 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

For an integer $n \geq 0$ find a closed form expression for the sum

$$\sum_{k=0}^{n}\frac{(-1)^{2^k}}{F_{3^{k+1}}(L_{3^k}L_{3^{k+1}}\cdots L_{3^n})^2}.$$

SOLUTIONS

The Number of Solutions of a Family of Boolean Equations

H-756 **Proposed by Russell J. Hendel, Towson University.**
(Vol. 52, No. 3, August 2014)

We seek to generalize a known problem that states that

$$\#\{\langle x_1,\ldots,x_{n+1}\rangle : x_1x_2 \vee x_2x_3 \vee \cdots \vee x_nx_{n+1} = 0\} = F_{n+3} \tag{1}$$

where x_i are Boolean variables for $i = 1,\ldots,n$. To generalize the above formula, we

(i) fix integers $d,\ i$ with $d > i \geq 1$;
(ii) let D_j be products of d Boolean variables x_k with consecutive indices such that D_j and D_{j+1} have i variables in common;
(iii) let m be the total number of variables occurring in $D_1,\ldots,D_n$ and
(iv) let
$$S_n = \#\{\langle x_1,\ldots,x_m\rangle : D_1 \vee D_2 \vee \cdots \vee D_n = 0\}.$$

Determine the coefficients of the minimal recursion satisfied by the $\{S_n\}_{n\geq 1}$.

Solution by the proposer.

Let us start with some examples.

Examples: To illustrate the notation we use (1):

$$d=2, i=1, D_j = x_jx_{j+1} \quad \text{and} \quad S_n = F_{n+3}.$$

The (minimal) recursion satisfied by the $\{S_n\}_{n\geq 1}$ is $S_{n+2} = S_{n+1} + S_n$.

For certain special cases of d and i, it is easy to discover the minimal recursions satisfied by $\{S_n\}_{n\geq 1}$. Some illustrative examples are as follows:

$$S_n = \left(2^{d-1}-1\right)S_{n-1} + 2^{d-2}S_{n-2}, \quad \text{for} \quad i=1 \quad \text{and arbitrary} \quad d.$$

$$S_n = \left(2^{i}-1\right)S_{n-1} + \left(2^{i}-1\right)S_{n-2}, \quad \text{for} \quad d=2i \quad \text{and arbitrary} \quad i.$$

$$S_n = \sum_{j=1}^{d} S_{n-j}, \quad \text{for} \quad i=d-1 \quad \text{and arbitrary} \quad d.$$

Now we are ready to proceed to the solution. With notations as presented in the proposed problem, we have the following theorem.

Theorem 1. (i) *Define k by $d = 2i + k$. If $k \geq 0$, then*

$$S_n = \left(2^{i+k} - 1\right) S_{n-1} + \left(2^{i+k} - 2^k\right) S_{n-2}. \tag{2}$$

(ii) *Define g, p, k by $g = d - i$ and $d = pg + k, 1 \leq k \leq g$. If $p \geq 1$,*

$$S_n = \sum_{u=1}^{p} \left(2^g - 1\right) S_{n-u} + \left(2^g - 2^{g-k}\right) S_{n-p-1}. \tag{3}$$

Since, the proofs of (2) and (3) are similar, it suffices to present the proof of (3). Prior to presenting the proof, we present some simple examples illustrating our notations.

Example 2. *In (3), let $d = 5, i = 3$. Then $g = 2, p = 2$ and $k = 1$.*
The equation $D_1 \vee D_2 \vee \cdots \vee D_n = 0$ becomes

$$x_1x_2x_3x_4x_5 \vee x_3x_4x_5x_6x_7 \vee \cdots \vee x_{2(n-1)+1} \cdots x_{2(n-1)+5} = 0.$$

Equation (3) asserts that the solutions satisfy the recursion $S_n = 3S_{n-1} + 3S_{n-2} + 2S_{n-3}$ for all $n \geq 4$.

Further numerical examples may be found in Table 1.

Table 1. The table presents numerical examples illustrating the main theorem. The top row lists the degree d while the left-most column lists i. For example row $i = 1$ and column $d = 2$ corresponds to the family of Boolean equations $x_1x_2 \vee x_2x_3 \vee \cdots \vee x_nx_{n+1} = 0$ where all disjuncts have degree $d = 2$ and every two consecutive disjuncts have $i = 1$ variables in common. Row $i = 1$ and column $d = 2$ contain the coefficients of the minimal recursion satisfied by the $\{S_n\}_{n\geq1}$, that is, $S_n = S_{n-1} + S_{n-2}$.

	$d = 2$	3	4	5	6	7	8
$i = 1$	$\langle 1, 1\rangle$	$\langle 3, 2\rangle$	$\langle 7, 4\rangle$	$\langle 15, 8\rangle$	$\langle 31, 16\rangle$	$\langle 63, 32\rangle$	$\langle 127, 64\rangle$
2		$\langle 1, 1, 1\rangle$	$\langle 3, 3\rangle$	$\langle 7, 6\rangle$	$\langle 15, 12\rangle$	$\langle 31, 24\rangle$	$\langle 63, 48\rangle$
3			$\langle 1, 1, 1, 1\rangle$	$\langle 3, 3, 2\rangle$	$\langle 7, 7\rangle$	$\langle 15, 14\rangle$	$\langle 31, 28\rangle$
4				$\langle 1, 1, 1, 1, 1\rangle$	$\langle 3, 3, 3\rangle$	$\langle 7, 7, 4\rangle$	$\langle 15, 15\rangle$
5					$\langle 1, 1, 1, 1, 1, 1\rangle$	$\langle 3, 3, 3, 2\rangle$	$\langle 7, 7, 6\rangle$

To prove (3), it will be convenient to only treat the case $k < g$ as the proof for the case $k = g$ is similar and omitted. Using this assumption it is easy to verify that

$$p = \left\lfloor \frac{d}{g} \right\rfloor, \qquad k = \left\langle \frac{d}{g} \right\rangle, \tag{4}$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x and $\langle d/g \rangle$ is the remainder of the division of d by g.

In the proof, we will use *word* terminology from semigroups. More specifically, we will speak about the prefix, factor or suffix of an elementary conjunction; for example, if discussing the elementary conjunction $x_2x_3x_4x_5$, x_2 is a prefix, x_5 is a suffix, and x_3x_4 is a factor. We will also interchange word and vector notation: e.g. we will say $\langle x_3, x_4\rangle$ is a factor of $x_2x_3x_4x_5$. We use boldface 1 to indicate the vector of all 1's, so that e.g. $\langle x_{j+1}, \ldots, x_{j+g}\rangle \neq \mathbf{1}$ means that not all g variables $x_{j+1}, \ldots, x_{j+g}$ are identically 1.

Figure 1, which facilitates the presentation of the proof, compactly summarizes the relationship between the indices of the Boolean variables and the disjuncts D_j.

The proof uses an induction argument. The two propositions below correspond to the base case and induction step.

Proposition 3.

$$\#\{\langle x_1, \ldots, x_m\rangle : \bigvee_{j=1}^{n} D_j = 0, \ \textit{with } \langle x_{m-g+1}, \ldots, x_m\rangle \neq \mathbf{1}\} = (2^g - 1)S_{n-1}.$$

Proof. The requirement that $\bigvee_{j=1}^{n} D_j = 0$, implies $D_n = 0$, which in turn requires that some variable occurring in D_n has value 0. There are $(2^g - 1)$ ways for $\langle x_{m-g+1}, \ldots, x_m\rangle \neq \mathbf{1}$. By definition of the S_n, there are S_{n-1} ways for the remaining $m-g$ variables to be solutions to $\bigvee_{j=1}^{n-1} D_j = 0$. □

$$\overbrace{\underbrace{x_{m-(\lfloor \frac{d}{g}\rfloor+2)g+1} \cdots x_{m-(\lfloor \frac{d}{g}\rfloor+1)g}}_{\text{last } g \text{ literals of } D_{n-(\lfloor \frac{d}{g}\rfloor+1)}}}^{\text{length } g \text{ suffix of } D_{n-(\lfloor \frac{d}{g}\rfloor+1)}} \overbrace{\underbrace{x_{m-(\lfloor \frac{d}{g}\rfloor+1)g+1} \cdots x_{m-(\lfloor \frac{d}{g}\rfloor+1)g+(g-\langle \frac{d}{g}\rangle)}}_{(g-\langle \frac{d}{g}\rangle) \text{ consecutive literals of } D_{n-\lfloor \frac{d}{g}\rfloor}}}^{\text{length } (g-\langle \frac{d}{g}\rangle) \text{ factor of } D_{n-\lfloor \frac{d}{g}\rfloor}}$$

$$\overbrace{\underbrace{x_{m-\lfloor \frac{d}{g}\rfloor g-\langle \frac{d}{g}\rangle+1} \cdots x_{m-\lfloor \frac{d}{g}\rfloor g}}_{\text{length } \langle \frac{d}{g}\rangle \text{ factor of } D_{n-q},\, 0 \leq q \leq \lfloor \frac{d}{g}\rfloor}}^{\text{length } \langle \frac{d}{g}\rangle \text{ prefix of } D_n}$$

$$\cdots \overbrace{\underbrace{x_{m-(q+1)g+1} \cdots x_{m-qg}}_{\text{length } g \text{ factor of } D_n}}^{\text{Suffix of } D_{n-q}} \overbrace{\underbrace{x_{m-qg+1} \cdots x_{m-(q-1)g}}_{g \text{ consecutive literals in } D_n}}^{\text{Suffix of } D_{n-(q-1)}} \cdots \overbrace{\underbrace{x_{m-g+1} \cdots x_m}_{\text{last } g \text{ literals in } D_n}}^{\text{Suffix of } D_n}$$

Figure 1. Illustration of the relation of the variables x_k to the D_j in (3).

Proposition 4. *Suppose that for some integer $q \geq 1$, we have*

$$\#\{\langle x_1, \ldots, x_m\rangle : {}_{j=1}^{n} \vee D_j = 0, \ \textit{with } \langle x_{m-qg+1}, \ldots, x_{m-(q-1)g}\rangle \neq \mathbf{1} \ \textit{and}$$
$$x_{m-u} \textit{ have arbitrary values for } 0 \leq u \leq (q-1)g - 1\} = \sum_{u=1}^{q} (2^g - 1)S_{n-u}. \quad (5)$$

If additionally,

$$(q+1)g < d, \quad (6)$$

then (5) *holds with q replaced by $q+1$.*

Proof. Replacing q with $q+1$ in the left-hand side of (5) necessitates assuming

$$\langle x_{m-(q+1)g+1}, \ldots, x_{m-qg}\rangle \neq \mathbf{1}, \quad (7)$$

and

$$x_{m-u} \text{ have arbitrary values for } 0 \leq u \leq qg-1. \tag{8}$$

By (6), $x_{m-(q+1)g+1} \cdots x_{m-qg}$ is a factor of D_n and hence (7) implies that $D_n = 0$.

There are $2^g - 1$ ways that (7) can take place. By the definition of the S_n, there are $S_{n-(q+1)}$ ways for the first $m-(q+1)g$ variables, $x_1, x_2, \ldots, x_{m-(q+1)g}$, to be solutions to $\bigvee_{j=1}^{n-(q+1)} D_j = 0$. Using an induction assumption, we conclude that the total number of solutions of $\bigvee_{j=1}^{n} D_j = 0$ with (8) and (7) holding is

$$(2^g - 1)S_{n-(q+1)} + \sum_{u=1}^{q}(2^g-1)S_{n-u} = \sum_{u=1}^{q+1}(2^g-1)S_{n-u}.$$

□

Corollary 5. *Using the notation in* (4)*, we have*

$$\#\{\langle x_1, \ldots, x_m\rangle : \bigvee_{j=1}^{n} D_j = 0, \text{ with } \langle x_{m-pg+1}, \ldots, x_{m-(p-1)g}\rangle \neq \mathbf{1} \text{ and}$$
$$x_{m-u} \text{ have arbitrary values } 0 \leq u \leq (p-1)g-1\} = \sum_{u=1}^{p}(2^g-1)S_{n-u}. \tag{9}$$

Proof. A routine induction argument with Proposition 3.1 as the base case and Proposition 3.2 as the induction step. □

Completion of the Proof of (3).

Proof. Assume

$$x_{m-u} = 1, \qquad 0 \leq u \leq \left\lfloor \frac{d}{g} \right\rfloor g - 1. \tag{10}$$

Since we require $D_n = 0$, this assumption requires that at least one of the $\langle \frac{d}{g} \rangle$ variables, $x_{m-\lfloor \frac{d}{g}\rfloor g - \langle \frac{d}{g}\rangle + 1}, \ldots, x_{m-\lfloor \frac{d}{g}\rfloor g}$ equals 0; that is, (10) implies

$$\langle x_{m-\lfloor \frac{d}{g}\rfloor g - \langle \frac{d}{g}\rangle + 1}, \ldots, x_{m-\lfloor \frac{d}{g}\rfloor g}\rangle \neq \mathbf{1}. \tag{11}$$

There are $2^{\langle \frac{d}{g}\rangle} - 1$ ways (11) can take place. Note that (11) also implies that $D_{n-q} = 0, 1 \leq q \leq \lfloor \frac{d}{g} \rfloor$, because the word on the left side of (11) is also a factor of $D_{n-q}, 1 \leq q \leq \lfloor \frac{d}{g} \rfloor$. Consequently, we are indifferent to the values of the $g - \langle \frac{d}{g} \rangle$ variables

$$x_{m-(\lfloor \frac{d}{g}\rfloor + 1)g+1}, \ldots, x_{m-(\lfloor \frac{d}{g}\rfloor+1)g+(g-\lfloor \frac{d}{g}\rfloor)}.$$

There are $2^{g-\langle \frac{d}{g}\rangle}$ ways this can happen. By the definition of the S_n, there are $S_{n-(\lfloor \frac{d}{g}+1\rfloor)}$ ways for the remaining $m - (\lfloor \frac{d}{g} \rfloor + 1)g$ variables to be solutions to $\bigvee_{j=1}^{n-(\lfloor \frac{d}{g}\rfloor+1)} D_j = 0$.

Hence, by (4), there are a total $(2^{\langle \frac{d}{g}\rangle} - 1)2^{g-\langle \frac{d}{g}\rangle} S_{n-(\lfloor \frac{d}{g}+1\rfloor)} = (2^k-1)2^{g-k}S_{n-(p+1)}$ solutions to $\bigvee_{j=1}^{n} D_j = 0$ with (10) and (11) holding.

The proof of (3) is completed by combining this case with (9). □

The Lucas Factorial of a Prime

H-757 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 52, No. 2, May 2014)

For an odd prime p prove that

$$\prod_{k=1}^{p} L_k \equiv \begin{cases} 2(-1)^{(p+1)/4} & \pmod{F_p} \quad \text{if} \quad p \equiv -1 \pmod 4, \\ (-1)^{(p-1)/4} F_{p-3} & \pmod{F_p} \quad \text{if} \quad p \equiv 1 \pmod 4. \end{cases}$$

Solution by the proposer.

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For $1 \le k \le p-1$, we have

$$\gcd(F_k, F_p) = F_{\gcd(k,p)} = F_1 = 1. \tag{1}$$

From Theorem 1 of [1], we have

$$\prod_{k=1}^{n} L_{2k} = \sum_{k=0}^{n} \binom{2n+1}{k}_F.$$

Letting $p = 2n+1$ in the above identity, by (1), we have

$$\prod_{k=1}^{(p-1)/2} L_{2k} = \sum_{k=0}^{(p-1)/2} \binom{p}{k}_F = 1 + \sum_{k=1}^{(p-1)/2} \frac{F_p F_{p-1} \cdots F_{p-k+1}}{F_k F_{k-1} \cdots F_1} \equiv 1 \pmod{F_p}. \tag{2}$$

From Theorem 3 of [1], we have

$$\prod_{k=1}^{n} L_{2k-1} = \sum_{k=0}^{2n} \mathbf{i}^{n-k} \binom{2n}{k}_F, \qquad \text{where} \qquad \mathbf{i} = \sqrt{-1}.$$

Letting $p = 2n-1$ in the above identity, by (1), we have

$$\begin{aligned} \prod_{k=1}^{(p+1)/2} L_{2k-1} &= \sum_{k=0}^{p+1} \mathbf{i}^{(p+1)/2-k} \binom{p+1}{k}_F \\ &= \mathbf{i}^{(p+1)/2} + \mathbf{i}^{-(p+1)/2} + \mathbf{i}^{(p-1)/2} F_{p+1} + \mathbf{i}^{-(p-1)/2} F_{p+1} \\ &\quad + \sum_{k=2}^{p-1} \mathbf{i}^{(p+1)/2-k} \times \frac{F_{p+1} F_p \cdots F_{p-k+2}}{F_k F_{k-1} \cdots F_1} \\ &\equiv (-1)^{(p+1)/4}(1 + (-1)^{(p+1)/2}) + (-1)^{(p-1)/4}(1 + (-1)^{(p-1)/2}) F_{p-1} \pmod{F_p}. \end{aligned}$$

So from the above, we immediately conclude that if $p \equiv -1 \pmod 4$, then

$$\prod_{k=1}^{(p+1)/2} L_{2k-1} \equiv 2(-1)^{(p+1)/4} \pmod{F_p}, \tag{3}$$

while if $p \equiv 1 \pmod 4$, then

$$\prod_{k=1}^{(p+1)/2} L_{2k-1} \equiv (-1)^{(p-1)/4} 2F_{p-1} \equiv (-1)^{(p-1)/4}(F_p + F_{p-3}) \equiv (-1)^{(p-1)/4} F_{p-3} \pmod{F_p}. \tag{4}$$

Since

$$\prod_{k=1}^{p} L_k = \prod_{k=1}^{(p-1)/2} L_{2k} \prod_{k=1}^{(p+1)/2} L_{2k-1},$$

the desired congruence follows from (2), (3) and (4).

References

[1] E. Kılıç, I. Akkuş, and H. Ohtsuka, *Some generalized Fibonomial sums related with the Gaussian q-binomial sums*, Bull. Math. Soc. Sci. Math. Roumanie Tome, **55.103** (2012), 51–61.

Limits of Factorials and Tangents at Exponents Involving Fibonacci Numbers

H-758 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Compute:

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}^{F_m}\left(\sqrt[n]{(2n-1)!!}^{F_{m+1}}\left(\tan\left(\frac{\pi(n+1)\sqrt[n+1]{n+1}}{4n\sqrt[n]{n}}\right)-1\right)^{F_{m+2}}\right)\right).$$

Solution by Ángel Plaza.

Since $F_{m+2} = F_{m+1} + F_{m+1}$ the proposed limit may be obtained by the product of the following two limits:

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}\left(\tan\left(\frac{\pi(n+1)\sqrt[n+1]{n+1}}{4n\sqrt[n]{n}}\right)-1\right)\right)^{F_m}, \quad \text{and}$$

$$\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\left(\tan\left(\frac{\pi(n+1)\sqrt[n+1]{n+1}}{4n\sqrt[n]{n}}\right)-1\right)\right)^{F_{m+1}}.$$

These limits are respectively equal to $\left(\frac{\pi}{2e}\right)^{F_m}$ and $\left(\frac{\pi}{e}\right)^{F_{m+1}}$ from where the result follows. Let us show that

$$\lim_{n\to\infty}\sqrt[n]{n!}\left(\tan\left(\frac{\pi(n+1)\sqrt[n+1]{n+1}}{4n\sqrt[n]{n}}\right)-1\right)=\frac{\pi}{2e}.$$

By Stirling's formula $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$, and $\lim_{x\to 1}\frac{\tan\left(\frac{\pi}{4}x\right)-1}{x-1}=\frac{\pi}{2}$. Then

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{n!}\left(\tan\left(\frac{\pi(n+1)\sqrt[n+1]{n+1}}{4n\sqrt[n]{n}}\right)-1\right) &= \frac{\pi}{2e}\lim_{n\to\infty} n\left(\frac{(n+1)\sqrt[n+1]{n+1}}{n\sqrt[n]{n}}-1\right)\\
&= \frac{\pi}{2e}\lim_{n\to\infty}\left((n+1)\sqrt[n+1]{n+1}-n\sqrt[n]{n}\right)\\
&= \frac{\pi}{2e}\lim_{n\to\infty}\frac{(n+1)\sqrt[n+1]{n+1}}{n}\text{(Stolz-Cezaro)}\\
&= \frac{\pi}{2e}.
\end{aligned}$$

The second limit may be obtained similarly taking into account that $(2n-1)!! = \frac{(2n)!}{2^n\, n!}$, and by Stirling's formula that $\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{2}{e}$.

Also solved by Dmitry Fleischman, Hideyuki Ohtsuka, and the proposers.

Late Acknowledgement. Kenneth B. Davenport solved H-755.

Errata. In H-783 (iii), the right-hand side of the equality to be proved is $\dfrac{35-15\sqrt{5}}{18}$ instead of $\dfrac{35-15\sqrt{3}}{18}$.