ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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PROBLEMS PROPOSED IN THIS ISSUE

H-689 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For positive integers l, m, and n such that $l \neq m$ prove that

$$F_m^n F_{ln} \equiv F_l^n F_{mn} \pmod{F_{m-l}}.$$

H-690 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let m and n be positive integers. Put

$$S_m(n) = \sum_{k=1}^{n} (-1)^{k(m+1)} F_k^{2m}.$$

Prove that

$$L_m S_m(n) = (-1)^{n(m+1)} S_1^m(n) - \sum_{i=1}^{\lfloor m/2 \rfloor} \sum_{r=1}^{i} \frac{m}{i} \binom{m-i-1}{i-1} \binom{i}{r} S_{m-r}(n).$$

H-691 Proposed by Ovidiu Furdui, Cluj, Romania and Huizeng Qin, Shandong, China

Find the value of

$$\sum_{n=1}^{\infty} (-1)^n \left(\ln 2 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} \right)^2.$$

H-692 Proposed by Napoleon Gauthier, Kingston, ON

Let $q \geq 1$, $N \geq 3$ be integers and define $Q = \lfloor (N-1)/2 \rfloor$. Find closed form expressions for the following sums:

a) $P_0(\theta, q) = \displaystyle\sum_{k=1}^{q} \frac{\sin(2k-1)\theta}{\cos^2 k\theta \cos^2(k-1)\theta}$;

b) $R_0(\theta,q)=\sum_{k=1}^{q}\frac{\sin(2k-1)\theta[\sin^2\theta+\sin^2(2k-1)\theta]}{\cos^4 k\theta\cos^4(k-1)\theta}$;

c) $P_1(N)=\sum_{k=1}^{Q}\frac{k\sin\frac{(2k-1)\pi}{N}}{\cos^2\frac{k\pi}{N}\cos^2\frac{(k-1)\pi}{N}}$;

d) $R_1(N)=\sum_{k=1}^{Q}\frac{k\sin\frac{(2k-1)\pi}{N}\left[\sin^2\frac{\pi}{N}+\sin^2\frac{(2k-1)\pi}{N}\right]}{\cos^4\frac{k\pi}{N}\cos^4\frac{(k-1)\pi}{N}}$.

SOLUTIONS

Cauchy-Schwartz to the Rescue

H-672 Proposed by J. L. Díaz-Barrero, Barcelona, Spain (Vol. 46, No. 2, May 2008)

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n}\left(\frac{F_k}{1+L_k}\right)^2\geq\frac{1}{F_nF_{n+1}}\left(\sum_{k=1}^{n}\frac{F_k^2}{1+L_k}\right)^2\geq F_n^3F_{n+1}^3\left(\sum_{k=1}^{n}F_k^2(1+L_k)\right)^{-2}.$$

Solution by Harris Kwong, Fredonia, NY

Both halves of the given inequality, in their equivalent forms, can be obtained from the Cauchy-Schwartz inequality and the identity $\sum_{k=1}^{n}F_k^2=F_nF_{n+1}$, as follows:

$$\left(\sum_{k=1}^{n}\frac{F_k^2}{1+L_k}\right)^2=\left(\sum_{k=1}^{n}F_k\cdot\frac{F_k}{1+L_k}\right)^2\leq\left(\sum_{k=1}^{n}F_k^2\right)\sum_{k=1}^{n}\left(\frac{F_k}{1+L_k}\right)^2=F_nF_{n+1}\sum_{k=1}^{n}\left(\frac{F_k}{1+L_k}\right)^2,$$

and

$$F_n^2F_{n+1}^2=\left(\sum_{k=1}^{n}F_k^2\right)^2=\left(\sum_{k=1}^{n}\frac{F_k}{\sqrt{1+L_k}}\cdot F_k\sqrt{1+L_k}\right)^2\leq\left(\sum_{k=1}^{n}\frac{F_k^2}{1+L_k}\right)\left(\sum_{k=1}^{n}F_k^2(1+L_k)\right).$$

Also solved by Paul S. Bruckman, Kenneth B. Davenport, H.-J. Seiffert, and the proposer.

Pell Numbers via Binomial Coefficients

H-673 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 46, No. 3, August 2008)

The Pell and Pell-Lucas numbers are defined by

$$P_0=0,\quad P_1=1,\quad\text{and}\quad P_{n+1}=2P_n+P_{n-1}\quad\text{for } n\geq 1,$$
$$Q_0=0,\quad Q_1=1,\quad\text{and}\quad Q_{n+1}=2Q_n+Q_{n-1}\quad\text{for } n\geq 1,$$

respectively. Prove that, for all positive integers n,

$$P_{2n-1} = 2^{-n}\sum_{k=0}^{2n-1}(-1)^{\lfloor(2n-5k-5)/4\rfloor}\binom{4n-1}{k},$$
$$Q_{2n} = 2^{1-n}\sum_{k=0}^{2n}(-1)^{\lfloor(2n-5k)/4\rfloor}\binom{4n+1}{k}.$$

Solution by the proposer

The Fibonacci polynomials are defined by $F_0(x)=0$, $F_1(x)=1$, and $F_{n+1}(x)=xF_n(x)+F_{n-1}(x)$ for all $n\geq 1$. It is well-known that

$$F_n(x)=\frac{\alpha(x)^n-\beta(x)^n}{\sqrt{x^2+4}}, \qquad n\geq 0, \tag{1}$$

where $\alpha(x)=(x+\sqrt{x^2+4})/2$ and $\beta(x)=(x-\sqrt{x^2+4})/2$. If $\gamma=\alpha(2)=1+\sqrt{2}$ and $\delta=\beta(2)=1-\sqrt{2}$, then

$$P_n=\frac{\gamma^n-\delta^n}{2\sqrt{2}} \qquad \text{and} \qquad Q_n=\gamma^n+\delta^n, \qquad n\geq 0. \tag{2}$$

From ([1], identity (1)), we know that

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}F_{2k+1}(x)=(x^2+4)^n, \qquad n\geq 0, \qquad x\in\mathbb{C}. \tag{3}$$

In (3), we take $x=i\sqrt{-\sqrt{2}\delta}=2i\sin(\pi/8)$, where $i=\sqrt{-1}$. Since $4+\sqrt{2}\delta=\sqrt{2}\gamma$, we get

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}a_k=(\sqrt{2}\gamma)^n, \qquad n\geq 0, \tag{4}$$

where

$$a_k:=F_{2k+1}(2i\sin(\pi/8))=\frac{\cos((2k+1)\pi/8)}{\cos(\pi/8)}, \qquad k\geq 0,$$

as is easily verified from (1). The sequence $(a_k)_{k\geq 0}$ satisfies the recurrence $a_{k+4}=-a_k$, $k\geq 0$, with initial values $a_0=1$, $a_1=\sqrt{2}-1$, $a_2=1-\sqrt{2}$, and $a_3=-1$. Now, it is easily seen that

$$a_k=(-1)^{\lfloor 5k/4\rfloor}+\sqrt{2}b_k, \qquad k\geq 0,$$

where the sequence $(b_k)_{k\geq 0}$ is defined as

$$b_k=\begin{cases}(-1)^{\lfloor(5k+4)/4\rfloor} & \text{if} \quad k\equiv 1,\ 2,\ 5,\ 6 \pmod 8;\\ 0 & \text{if} \quad k\equiv 0,\ 3,\ 4,\ 7 \pmod 8.\end{cases}$$

In (4), replace n by $2n-1$. Since $\gamma^{2n-1}=(Q_{2n-1}+2\sqrt{2}P_{2n-1})/2$, as is seen from (2), and since $\sqrt{2}$ is irrational, after reindexing k by $2n-1-k$, we obtain the first proposed identity and also

$$Q_{2n-1}=2^{2-n}\sum_{k=0}^{2n-1}b_{2n-1-k}\binom{4n-1}{k}.$$

Similarly, (4) with n replaced by $2n$ gives the second proposed identity and also

$$P_{2n}=2^{-n}\sum_{k=0}^{2n}b_{2n-k}\binom{4n+1}{k}.$$

References

[1] H.-J. Seiffert, *Solution to Problem H-591*, The Fibonacci Quarterly, **41.5** (2003), 473–475.

Also solved by Paul S. Bruckman.

An Inequality Involving π and Fibonacci Numbers

H-674 Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania (Vol. 46, No. 3, August 2008)

Let n be a positive integer. Prove that

$$n\pi^2 F_n F_{n+1} \leq (n(F_n - 1) + \pi(F_{n+2} - 1))^2.$$

Editor's solution

Let $\alpha = (1+\sqrt{5})/2$. We use the well-known fact (easily proved by induction on k) that $\alpha^{k-2} < F_k < \alpha^{k-1}$ for all $k \geq 3$. We have

$$F_{n+2} - 1 > F_{n+1} > \alpha^{n-1} > n \qquad \text{for all} \quad n \geq 4.$$

Thus, on the right-hand side we have

$$n(F_n - 1) + \pi(F_{n+2} - 1) > nF_n - n + (F_{n+2} - 1) > nF_n.$$

Hence, for $n \geq 4$ it suffices to prove the inequality obtained by replacing the right-hand side of it with $(nF_n)^2$. Now

$$(nF_n)^2 > (n\alpha^{n-2})^2 = n^2\alpha^{2n-4},$$

while

$$n\pi^2 F_n F_{n+1} < n\pi^2\alpha^{n-1}\alpha^n = n\pi^2\alpha^{2n-1}.$$

Thus, it suffices that

$$n\pi^2\alpha^{2n-1} < n^2\alpha^{2n-4},$$

or $n > \pi^2\alpha^3$, which holds for $n \geq 42$. The desired inequality can be now be checked by hand for all the smaller values of n.

Also solved by Paul S Bruckman, Kenneth B. Davenport, and the proposers.

Perfect Primes

H-675 Proposed by John J. Jaroma, Ave Maria, Florida (Vol. 46, No. 3, August 2008)

An odd perfect number is an odd integer that is equal to the sum of its proper divisors. Although such a number is currently unknown, many conditions necessary for its existence have been established. The earliest is attributed to Euler who showed that if n is an odd perfect number then

$$n = p^\alpha p_1^{2\beta_1} \cdots p_r^{2\beta_r}$$

where $p, p_1, \ldots, p_r$ are distinct odd primes and $p \equiv \alpha \equiv 1 \pmod 4$. The prime p has been dubbed the *special prime*. Show that the least prime divisor of n is not p.

Solution by Douglas Iannucci, St. Thomas, VI

Since n is perfect, we have $\sigma(p^\alpha) \mid 2n$. Write $\alpha = 4k+1$ for some k. Then

$$\sigma(p^\alpha) = \frac{p^{4k+2}-1}{p-1} = \frac{p^{2k+1}-1}{p-1} \cdot \frac{p^{2k+1}+1}{p+1} \cdot (p+1).$$

Since $(p^{2k+1} \pm 1)/(p \pm 1)$ are both integers, we have $p+1 \mid 2n$; therefore

$$\frac{p+1}{2} \mid n.$$

Since $(p+1)/2$ is an odd integer (greater than 1), it is divisible by an odd prime, say q; clearly $q < p$ (since $(p+1)/2 < p$). Therefore, p is not the least prime divisor of n.

Also solved by Paul S. Bruckman and the proposer.

A Wicked Composition

H-676 Proposed by Mohammad K. Azarian, Evansville, Indiana

Let $f(x) = \sinh x$, $g(x) = \ln(x+\sqrt{1+x^2})$, and $h(x) = 1/(2-f(-g(-x)))$. Also, let $h_0(x) = h(x)$, $h_1(x) = h_0(h_0(x)), \ldots$ and $h_{n+1}(x) = h_0(h_n(x))$ for all $n \geq 0$. If $p(x) = \prod_{i=0}^{n} h_i(x)$, then find the coefficient of F_k^r in the expansion of $1/\sqrt{p(F_k)}$ in terms of r and n.

Solution by Paul S. Bruckman

Note that $-g(-x) = -\ln(-x+\sqrt{1+x^2}) = \ln(x+\sqrt{1+x^2}) = g(x)$. Hence,

$$\begin{aligned} f(-g(-x)) &= f(g(x)) = \sinh\left(\ln(x+\sqrt{1+x^2})\right) \\ &= \frac{1}{2}\left(x+\sqrt{1+x^2}) - (-x+\sqrt{1+x^2})\right) = x. \end{aligned}$$

Then

- $h(x) = h_0(x) = h_0 = 1/(2-x)$;
- $h_1(x) = h_1 = 1/(2-h_0) = (2-x)/(3-2x)$;
- $h_2(x) = h_2 = 1/(2-h_1) = (3-2x)/(4-3x)$;

etc. By an easy induction process on n, we find that

$$h_n(x) = \frac{(n+1)-nx}{(n+2)-(n+1)x} \qquad \text{for all} \quad n \geq 0.$$

It then follows that $p(x) = ((n+2)-(n+1)x)^{-1}$. Hence,

$$\frac{1}{\sqrt{p(F_k)}} = ((n+2)-(n+1)F_k)^{1/2} = (n+2)^{1/2}\sqrt{1-\frac{(n+1)F_k}{n+2}} := A(n,k).$$

Note that $A(n,k)$ is imaginary for $k \geq 3$ and $n > 0$. Nevertheless, proceeding naively by expanding the square root, we get

$$A(n,k) = (n+2)^{1/2}\sum_{r=0}^{\infty}(-1)^r\binom{1/2}{r}\left(\frac{(n+1)F_k}{n+2}\right)^r.$$

The desired coefficient is therefore equal to

$$(-1)^r(n+2)^{1/2}\binom{1/2}{r}\left(\frac{n+1}{n+2}\right)^r = -(n+1)^r(n+2)^{1/2-r}2^{2r-4}(2r-1)^{-1}\binom{2r}{r}.$$

Also solved by Kenneth B. Davenport and the proposer.

Late Acknowledgement. G. C. Greubel solved part a) of H-670.

Errata. In H-685 part d), the right-hand side should be "F_k" instead of "F_{3k}".