ADVANCED PROBLEMS AND SOLUTIONS

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A TRIBUTE TO PAUL S. BRUCKMAN

Contributed by
Napoleon Gauthier and Florian Luca

This issue of the Advanced Problems and Solutions (APS) section is dedicated to Paul Bruckman, in recognition of his forty years of significant contributions to *The Fibonacci Quarterly.* Paul is a Sustaining Member of The Fibonacci Association and his support for, and loyalty to, the *FQ* is worthy of mention. Paul published his first *FQ* paper in 1972. In that same year, Paul also began solving virtually all the problems proposed in the EPS (E:Elementary) and in the APS sections of *FQ*. From 1972 to the present, Paul has published 20 articles in *FQ*. He is also to be credited with 35 proposals and in excess of 750 solutions in the EPS section of the journal, in parallel with 82 proposals and in excess of 310 solutions in the APS section. Paul's enthusiasm for *FQ* has never waned and his name is now an integral part of the history and lore of *FQ*.

PROBLEMS PROPOSED IN THIS ISSUE

H-704 Proposed by Paul S. Bruckman, Nanaimo, Canada

Prove the following identity:

$$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n-2k}{2k}2^{n+1-4k}=P_{n+1}+n+1,$$

where $\{P_n\}_{n\geq 0}$ is the ordinary Pell sequence.

H-705 Proposed by Paul S. Bruckman, Nanaimo, Canada

Define the following sum

$$S_n(a,b)=\sum_{k=0}^{\lfloor (n-b)/a\rfloor}\binom{n}{ak+b},$$

where n, a and b are integers with $0 \le b < a \le n$. Prove the following relation: $S_{am+2b}(a,b) = 2S_{am+2b-1}(a,b)$, $m = 1, 2, \ldots$.

H-706 **Proposed by Paul S. Bruckman, Nanaimo, Canada**

Define the following sum:

$$S_n = \frac{1}{2}\left(\sum_{k=n+1}^{3n} \frac{1}{k^2 - n^2}\right)^{-1}.$$

Show that $S(n) \sim \pi(n)$ as $n \to \infty$, where $\pi(n)$ is the counting function of the primes $p \le n$.

H-707 **Proposed by Paul S. Bruckman, Nanaimo, Canada**

Write $[P] = [a_1, a_2, \ldots, a_n]$, where $a_k = a_{n+1-k}$, $k = 1, 2, \ldots, n$; then $[P]$ is a palindromic simple continued fraction (scf); here, the a_k's are positive integers. Also, write $[0, P^*] = [0, a_1, \ldots, a_{n-1}]$. Finally, let $[\overline{P}]$ denote the infinite periodic scf $[P, P, P, \ldots]$. Prove the following: $[\overline{P}] - [0, \overline{P}] = [P] - [0, P^*]$.

SOLUTIONS

Binomial Sums With Fibonacci Numbers

H-685 **Proposed by N. Gauthier, Kingston, ON**
(Vol. 47, No. 1, February 2009/2010)

For k a positive integer prove the following identities:

a) $\displaystyle\sum_{m=1}^{k} \binom{2k-m-1}{k-1}(F_{2m} + F_m) = F_{3k}$;

b) $\displaystyle\sum_{m=1}^{k} \frac{m}{k}\binom{2k-m-1}{k-1}(F_{2m+2} - F_{m+1}) = F_{3k}$;

c) $\displaystyle\sum_{m=1}^{k} \frac{2^m}{2^{2k}}\binom{2k-m-1}{k-1}(F_{2m} + (-1)^{m+1}F_m) = F_k$;

d) $\displaystyle\sum_{m=1}^{k} \frac{2^m m}{2^{2k+1}k}\binom{2k-m-1}{k-1}(F_{2m+2} + (-1)^{m+1}F_{m+1}) = F_{3k}$.

Solution by Paul Bruckman

Let

$$U_k(x) = \sum_{m=1}^{k} \binom{2k-m-1}{k-1} x^m, \quad k = 1, 2, \ldots, \quad \text{with} \quad U_0(x) = 0. \tag{1}$$

Note that

$$U_k(x) = \sum_{m=0}^{k-1} \binom{m+k-1}{m} x^{k-m}. \tag{2}$$

We prove the following recurrence relation

$$(x-1)U_k(x) - x^2 U_{k-1}(x) = \frac{x^2}{2}\delta_{k,1} + \frac{1}{2}\binom{2k-2}{k-1} x(x-2), \quad k = 1, 2, \ldots, \tag{3}$$

where $\delta_{a,b}$ is the Kronecker δ-function which equals 1 if $a = b$ and 0, otherwise.

First, if $k=1$, we have $U_1(x)=x$. The recurrence in (3) yields:

$$(x-1)U_1(x)=\frac{x^2}{2}+\frac{1}{2}x(x-2)=x^2-x=x(x-1),$$

so $U_1(x)=x$, the correct value.

Now assume that $k\ge 2$. The left side of (3) becomes

$$\begin{aligned}
&\sum_{m=0}^{k-1}\binom{m+k-1}{m}x^{k+1-m}-\sum_{m=0}^{k-1}\binom{m+k-1}{m}x^{k-m}-\sum_{m=0}^{k-2}\binom{m+k-2}{m}x^{k+1-m}\\
&=\sum_{m=0}^{k-1}\binom{m+k-1}{m}x^{k+1-m}-\sum_{m=1}^{k}\binom{m+k-2}{m-1}x^{k+1-m}-\sum_{m=0}^{k-2}\binom{m+k-2}{m}x^{k+1-m}\\
&=x^{k+1}-x^{k+1}+\binom{2k-2}{k-1}x^2-\binom{2k-2}{k-1}x-\binom{2k-3}{k-2}x^2\\
&\quad+\sum_{m=1}^{k-2}x^{k+1-m}\left(\binom{m+k-1}{m}-\binom{m+k-2}{m-1}-\binom{m+k-2}{m}\right)\\
&=0+\binom{2k-3}{k-1}x^2-\binom{2k-2}{k-1}x+0=\frac{1}{2}\binom{2k-2}{k-1}(x^2-2x),
\end{aligned}$$

which completes the proof of (3).

Proof of a). Let

$$S_k=\sum_{m=1}^{k}\binom{2k-m-1}{k-1}(F_{2m}+F_m). \tag{4}$$

Then

$$S_k=\frac{1}{\sqrt{5}}\left(U_k(\alpha^2)-U_k(\beta^2)+U_k(\alpha)-U_k(\beta)\right), \tag{5}$$

by the Binet formula for F_n, where $(\alpha,\beta)=((1+\sqrt{5})/2,(1-\sqrt{5})/2)$. Now substituting $x=\alpha$ in (3) gives $U_1(\alpha)=\alpha$. Also, if $k\ge 2$,

$$\frac{1}{\alpha}U_k(\alpha)=\alpha^2U_{k-1}(\alpha)+\frac{1}{2}\binom{2k-2}{k-1}\beta;\quad\text{or}$$

$$U_k(\alpha)=\alpha^3U_{k-1}(\alpha)-\frac{1}{2}\binom{2k-2}{k-1}\quad\text{and likewise}\quad U_k(\beta)=\beta^3U_{k-1}(\beta)-\frac{1}{2}\binom{2k-2}{k-1}. \tag{6}$$

Therefore, by an easy induction

$$U_k(\alpha)=\alpha^{3k-2}-\frac{1}{2}\sum_{j=1}^{k-1}\alpha^{3(k-1-j)}\binom{2j}{j}\quad\text{and}\quad U_k(\beta)=\beta^{3k-2}-\frac{1}{2}\sum_{j=1}^{k-1}\beta^{3(k-1-j)}\binom{2j}{j}. \tag{7}$$

Then

$$\frac{1}{\sqrt{5}}(U_k(\alpha)-U_k(\beta))=F_{3k-2}-\frac{1}{2}\sum_{j=1}^{k-1}\binom{2j}{j}F_{3(k-1-j)}. \tag{8}$$

Now setting $x=\alpha^2$ in (3) yields the following recurrence

$$U_k(\alpha^2)=\alpha^3U_{k-1}(\alpha^2)+\frac{1}{2}\binom{2k-2}{k-1}\quad\text{and also}\quad U_k(\beta^2)=\beta^3U_{k-1}(\beta^2)+\frac{1}{2}\binom{2k-2}{k-1}. \tag{9}$$

Note that $U_1(\alpha^2) = \alpha^2$. Again, induction yields the following

$$U_k(\alpha^2) = \alpha^{3k-1} + \frac{1}{2}\sum_{j=1}^{k-1} \alpha^{3(k-1-j)}\binom{2j}{j} \quad \text{and} \quad U_k(\beta^2) = \beta^{3k-1} + \frac{1}{2}\sum_{j=1}^{k-1} \beta^{3(k-1-j)}\binom{2j}{j}. \tag{10}$$

Then

$$\frac{1}{\sqrt{5}}(U_k(\alpha^2) - U_k(\beta^2)) = F_{3k-1} + \frac{1}{2}\sum_{j=1}^{k-1}\binom{2j}{j}F_{3(k-1-j)}. \tag{11}$$

Now combining (8) and (11), we see from (5) that $S_k = F_{3k-2} + F_{3k-1} = F_{3k}$.

Proof of b). Let

$$V_k(x) = \sum_{m=1}^{k} \frac{m}{k}\binom{2k-m-1}{k-1}x^m, \quad k = 1, 2, \ldots, \quad \text{with} \quad V_0(x) = 0. \tag{12}$$

Then

$$V_k(x) = \sum_{m=0}^{k-1}\binom{k+m-1}{k-1}\left(\frac{k-m}{m}\right)x^{k-m} = \sum_{m=0}^{k-1}\binom{k+m-1}{k-1}x^{k-m} - \sum_{m=1}^{k-1}\binom{k+m-1}{k}x^{k-m}.$$

Since

$$U_k(x) = \sum_{m=0}^{k-1}\binom{k+m-1}{k-1}x^{k-m} = \sum_{m=0}^{k-1}\binom{k+m-1}{m}x^{k-m},$$

we find that for $k > 0$ and $x \neq 0$,

$$V_k(x) = U_k(x) - \frac{U_{k+1}(x)}{x^2} + \frac{1}{x}\binom{2k}{k} + \binom{2k-1}{k}. \tag{13}$$

Now let

$$T_k = \sum_{m=1}^{k}\frac{m}{k}\binom{2k-m-1}{k-1}(F_{2m+2} - F_{m+1}). \tag{14}$$

We then see that

$$T_k = \frac{1}{\sqrt{5}}\left(\alpha^2 V_k(\alpha^2) - \beta^2 V_k(\beta^2) - \alpha V_k(\alpha) + \beta V_k(\beta)\right). \tag{15}$$

We return to (13) and substitute the results of (7). Then

$$V_k(\alpha) = U_k(\alpha) - \beta^2 U_{k+1}(\alpha) - \beta\binom{2k}{k} + \binom{2k-1}{k} = \alpha^{3k-2} - \frac{1}{2}\sum_{j=1}^{k-1}\alpha^{3(k-1-j)}\binom{2j}{j}$$
$$- \beta^2\alpha^{3k+1} + \frac{\beta^2}{2}\sum_{j=1}^{k}\alpha^{3(k-j)}\binom{2j}{j} - \beta\binom{2k}{k} + \binom{2k-1}{k},$$

or, after simplification,

$$V_k(\alpha) = -\alpha^{3k-3} - \frac{\beta}{2}\sum_{j=1}^{k-1}\alpha^{3(k-1-j)}\binom{2j}{j} + \frac{\alpha}{2}\binom{2k}{k} + \binom{2k-1}{k}. \tag{16}$$

Similarly,

$$V_k(\beta) = -\beta^{3k-3} - \frac{\alpha}{2}\sum_{j=1}^{k-1}\beta^{3(k-1-j)}\binom{2j}{j} + \frac{\beta}{2}\binom{2k}{k} + \binom{2k-1}{k}. \tag{17}$$

Also, in (13), we substitute the results of (10) and obtain, after simplification,

$$V_k(\alpha^2) = \alpha^{3k-3} + \frac{\beta^2}{2}\sum_{j=1}^{k-1}\alpha^{3(k-1-j)}\binom{2j}{j} - \frac{\beta}{2}\binom{2k}{k} + \binom{2k-1}{k}. \tag{18}$$

Similarly,

$$V_k(\beta^2) = \beta^{3k-3} + \frac{\alpha^2}{2}\sum_{j=1}^{k-1}\beta^{3(k-1-j)}\binom{2j}{j} - \frac{\alpha}{2}\binom{2k}{k} + \binom{2k-1}{k}. \tag{19}$$

Now, combining the results in (16)–(19), the formula in (15) shows that $T_k\sqrt{5}$ equals

$$\alpha^{3k-1} - \beta^{3k-1} + \frac{1}{2}\sum_{j=1}^{k-1}\left(\alpha^{3(k-1-j)} - \beta^{3(k-1-j)}\right)\binom{2j}{j} + (\alpha-\beta)\binom{2k}{k} + (\alpha^2-\beta^2)\binom{2k-1}{k} +$$

$$\alpha^{3k-2} - \beta^{3k-2} - \frac{1}{2}\sum_{j=1}^{k-1}\left(\alpha^{3(k-1-j)} - \beta^{3(k-1-j)}\right)\binom{2j}{j} - (\alpha^2-\beta^2)\binom{2k}{k} + (\alpha-\beta)\binom{2k-1}{k},$$

or

$$T_k = F_{3k-1} + F_{3k-2} + (F_1 - F_2)\left(\binom{2k}{k} - \binom{2k-1}{k}\right) = F_{3k}.$$

Proof of c). Let

$$W_k(x) = \sum_{m=1}^{k}\frac{2^m}{2^{2k}}\binom{2k-m-1}{k-1}x^m, \quad k = 1, 2, \ldots, \quad \text{with} \quad W_0(x) = 0. \tag{20}$$

We note that

$$W_k(x) = \frac{U_k(2x)}{4^k}. \tag{21}$$

The recurrence in (3) is then transformed as follows:

$$(2x-1)W_k(x) = x^2W_{k-1}(x) + \frac{x}{2^{2k-1}}\left(x\delta_{k,1} + (x-1)\binom{2k-2}{k-1}\right). \tag{22}$$

Next, let

$$Y_k = \sum_{m=1}^{k}\frac{2^m}{2^{2k}}\binom{2k-m-1}{k-1}\left(F_{2m} + (-1)^{m+1}F_m\right). \tag{23}$$

We see that

$$Y_k = \frac{1}{\sqrt{5}}\left(W_k(\alpha^2) - W_k(\beta^2) - W_k(-\alpha) + W_k(-\beta)\right). \tag{24}$$

Setting $x = -\alpha$ in (22), we obtain

$$-\alpha^3W_k(-\alpha) = \alpha^2W_{k-1}(-\alpha) + \frac{\alpha^2}{2^{2k-1}}\left(\delta_{k,1} + \alpha\binom{2k-2}{k-1}\right),$$

or

$$W_k(-\alpha) = \beta W_{k-1}(-\alpha) + \frac{1}{2^{2k-1}}\left(\beta\delta_{k,1} - \binom{2k-2}{k-1}\right). \tag{25}$$

Similarly, setting $x = -\beta$ in (22) yields

$$W_k(-\beta) = \alpha W_{k-1}(-\beta) + \frac{1}{2^{2k-1}}\left(\alpha\delta_{k,1} - \binom{2k-2}{k-1}\right). \tag{26}$$

Likewise, setting $x = \alpha^2$ in (22) yields:

$$\alpha^3 W_k(\alpha^2) = \alpha^4 W_{k-1}(\alpha^2) + \frac{\alpha^3}{2^{2k-1}}\left(\alpha\delta_{k,1} + \binom{2k-2}{k-1}\right),$$

or

$$W_k(\alpha^2) = \alpha W_{k-1}(\alpha^2) + \frac{1}{2^{2k-1}}\left(\alpha\delta_{k,1} + \binom{2k-2}{k-1}\right). \tag{27}$$

Similarly, setting $x = \beta^2$ in (22) yields

$$W_k(\beta^2) = \beta W_{k-1}(\beta^2) + \frac{1}{2^{2k-1}}\left(\beta\delta_{k,1} + \binom{2k-2}{k-1}\right). \tag{28}$$

Induction on (25) and (26), respectively, leads to the following results for $k > 0$:

$$W_k(-\alpha) = \frac{1}{2}\beta^{k-2} - \sum_{j=1}^{k-1}\frac{\beta^{k-1-j}}{2^{2j+1}}\binom{2j}{j}; \quad W_k(-\beta) = \frac{1}{2}\alpha^{k-2} - \sum_{j=1}^{k-1}\frac{\alpha^{k-1-j}}{2^{2j+1}}\binom{2j}{j}. \tag{29}$$

Also, induction on (27) and (28), respectively, leads to the following results for $k > 0$;

$$W_k(\alpha^2) = \frac{1}{2}\alpha^{k+1} + \sum_{j=1}^{k-1}\frac{\alpha^{k-1-j}}{2^{2j+1}}\binom{2j}{j}; \quad W_k(\beta^2) = \frac{1}{2}\beta^{k+1} + \sum_{j=1}^{k-1}\frac{\beta^{k-1-j}}{2^{2j+1}}\binom{2j}{j}. \tag{30}$$

Combining the results of (29) and (30) into the formula (24) yields:

$$Y_k = \frac{1}{2}F_{k+1} + \sum_{j=1}^{k-1}\frac{F_{k-1-j}}{2^{2j+1}}\binom{2j}{j} + \frac{1}{2}F_{k-2} - \sum_{j=1}^{k-1}\frac{F_{k-1-j}}{2^{2j+1}}\binom{2j}{j} = F_k.$$

Proof of d). Let

$$Z_k(x) = \sum_{m=1}^{k}\frac{2^m m}{2^{2k+1}k}\binom{2k-m-1}{k-1}x^m, \quad k = 1, 2, \dots, \quad \text{with} \quad Z_0(x) = 0. \tag{31}$$

We note that

$$Z_k(x) = \frac{V_k(2x)}{2^{2k+1}}. \tag{32}$$

Returning to (13), expressing $V_k(x)$ in terms of $U_k(x)$ and also using (3), we may verify, after some simplifications, that $V_k(x)$ satisfies the following recurrence:

$$(x-1)V_k(x) - x^2 V_{k-1}(x) = -2x\binom{2k-2}{k-1} + \frac{x}{2}\binom{2k}{k}, \quad k = 2, 3, \dots, \quad \text{and} \quad V_1(x) = x. \tag{33}$$

Setting $x = -2\alpha$ in (33) yields the following recurrence valid for all $k \ge 2$:

$$V_k(-2\alpha) = 4\beta V_{k-1}(-2\alpha) - 4\beta^2\binom{2k-2}{k-1} + \beta^2\binom{2k}{k}. \tag{34}$$

Also, $V_1(-2\alpha) = -2\alpha$. Then, by induction on (34), we obtain

$$V_k(-2\alpha) = -4^k\beta^{k-1} + \beta\binom{2k}{k} + \sum_{j=1}^{k}(4\beta)^{k-j}\binom{2j}{j}. \tag{35}$$

Likewise,

$$V_k(-2\beta) = -4^k\alpha^{k-1} + \alpha\binom{2k}{k} + \sum_{j=1}^{k}(4\alpha)^{k-j}\binom{2j}{j}. \tag{36}$$

Also, setting $x = 2\alpha^2$ in (33) yields the recurrence

$$V_k(2\alpha^2) = 4\alpha V_{k-1}(2\alpha^2) + 4\beta\binom{2k-2}{k-1} - \beta\binom{2k}{k}. \tag{37}$$

Similarly,

$$V_k(2\beta^2) = 4\beta V_{k-1}(2\beta^2) + 4\alpha\binom{2k-2}{k-1} - \alpha\binom{2k}{k}. \tag{38}$$

Induction on (37) yields

$$V_k(2\alpha^2) = 4^k\alpha^{k-1} + \beta^2\binom{2k}{k} - \beta^3\sum_{j=1}^{k}(4\alpha)^{k-j}\binom{2j}{j}. \tag{39}$$

Likewise

$$V_k(2\beta^2) = 4^k\beta^{k-1} + \alpha^2\binom{2k}{k} - \alpha^3\sum_{j=1}^{k}(4\beta)^{k-j}\binom{2j}{j}. \tag{40}$$

Now make the following definition:

$$X_k = \sum_{m=1}^{k}\frac{2^m m}{2^{2k+1}k}\binom{2k-m-1}{k-1}\left(F_{2m+2} + (-1)^{m+1}F_{m+1}\right). \tag{41}$$

We see that

$$X_k = \frac{1}{\sqrt{5}}\left(\alpha^2 Z_k(\alpha^2) - \beta^2 Z_k(\beta^2) - \alpha Z_k(-\alpha) + \beta Z_k(-\beta)\right). \tag{42}$$

In light of (32), the following is also true

$$2^{2k+1}X_k\sqrt{5} = \alpha^2 V_k(2\alpha^2) - \beta^2 V_k(2\beta^2) - \alpha V_k(-2\alpha) + \beta V_k(-2\beta). \tag{43}$$

Substituting the results of (35)–(36) and (39)–(40) into (43) yields the following, after considerable cancellation of terms and other simplification:

$$2^{2k+1}X_k = 4^k(F_{k+1} + F_{k-2}) = 2^{2k+1}F_k, \quad \text{or} \quad X_k = F_k.$$

This completes, and also corrects, the indicated result for Part d).

Also solved by Eduardo H. M. Brietzke and the proposer.

Summing Products of Fibonacci Numbers

H-686 Proposed by José Luis Díaz-Barero, Barcelona, Spain (Vol. 47, No. 1, February 2009/2010)

Let n be a positive integer. Compute

$$\sum_{1\le i<j\le n} F_iF_j(F_i - F_j)^2.$$

Solution by Paul Bruckman

Let S_k denote $\sum_{i=1}^{n} F_i^k$ for $k = 1, 2, 3, 4$, where we treat n as fixed. If T denotes $\sum_{1 \le i < j \le n} F_i F_j (F_i - F_j)^2$, we see that

$$\begin{aligned} T &= \sum_{1 \le i < j \le n} F_i^3 F_j - 2 \sum_{1 \le i < j \le} F_i^2 F_j^2 + \sum_{1 \le i < j \le n} F_i F_j^3 \\ &= \sum_{1 \le i \ne j \le n} F_i^3 F_j - 2 \sum_{1 \le i < j \le n} F_i^2 F_j^2 \\ &= S_1 S_3 - S_4 - (S_2^2 - S_4) = S_1 S_3 - S_2^2. \end{aligned}$$

Now the following results are well-known

$$S_1 = F_{n+2} - 1, \qquad S_2 = F_n F_{n+1}.$$

A lesser known result which may be derived from the Binet formula is

$$S_3 = \frac{1}{10} \left(F_{3n+2} - 6(-1)^n F_{n-1} + 5 \right).$$

It appears in [1]. An alternative and equivalent formulation is the following:

$$S_3 = \frac{1}{2} \left(F_n F_{n+1}^2 - (-1)^n F_{n-1} + 1 \right).$$

Therefore,

$$\begin{aligned} T &= (F_{n+2} - 1) \frac{1}{2} (F_n F_{n+1}^2 - (-1)^n F_{n-1} + 1) - (F_n F_{n+1})^2 \\ &= \frac{1}{2} \left(F_{n+2} F_{n+1}^2 F_n - (-1)^n F_{n+2} F_{n-1} + F_{n+2} - F_n F_{n+1}^2 + (-1)^n F_{n-1} - 1 - 2F_n^2 F_{n+1}^2 \right) \\ &= \frac{1}{2} \left(F_{n+1}^2 F_n F_{n-1} - (-1)^n F_{n+2} F_{n-1} + F_{n+2} - F_n F_{n+1}^2 + (-1)^n F_{n-1} - 1 \right). \end{aligned}$$

References

[1] C. Cooper and R. E. Kennedy, *Problem 3*, Missouri Journal of Mathematical Sciences, **0.1** (1988), 29; *Solution*, ibid., **1.2** (1989), 31–32.

Also solved by the proposer.