ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2012. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

<u>B-1091</u> Proposed by Paul S. Bruckman, Nanaimo, BC, Canada.

If

$$S_n = \left(\sum_{k=n}^{\infty} \frac{1}{F_k}\right)^{-1},$$

show that $S_n \sim \frac{\alpha^{n-2}}{\sqrt{5}}$ as $n \to \infty$.

B-1092 **Proposed by José Luis Díaz-Barrero, Technical University of Catalonia, Barcelona, Spain.**

Compute the sum

$$\sum_{n=0}^{\infty} \frac{1}{(5\alpha)^n(n+2)} \sum_{k=0}^{n} \frac{F_{k+1}F_{n-k+1}}{k+1}.$$

B-1093 **Proposed by Paul S. Bruckman, Nanaimo, BC, Canada.**

Prove the following identity:

$$\sum_{k=0}^{[n/3]} (-1)^k \binom{n-2k}{k} 2^{n-3k} = F_{n+3} - 1.$$

B-1094 **Proposed by Paul S. Bruckman, Nanaimo, BC, Canada.**

Prove the following identity:

$$\sum_{k=0}^{[n/3]} \binom{n-k}{2k} 2^{n-3k} = F_{2n} + 1.$$

B-1095 **Proposed by Sergio Falcón and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.**

For any positive integer k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$, with initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$. Let $\{a_{n,j}\}$ be the integer matrix defined by $a_{n,j} = \binom{n}{j} - k\binom{n}{j+1} - \binom{n}{j+2}$. Prove that

$$\sum_{j=0}^{n} a_{n,j} F_{k,j+1} = 1.$$

SOLUTIONS

Higher Powers Equalities

B-1071 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 48.3, August 2010)

Prove the following identities:

(1) $F_{n-1}^4 + 4F_n^4 + 4F_{n+1}^4 + F_{n+2}^4 = 6F_{2n+1}^2$,

(2) $F_{n-1}^6 + 8F_n^6 + 8F_{n+1}^6 + F_{n+2}^6 = 10F_{2n+1}^3$.

Solution I by Paul S. Bruckman, 38 Front St., Unit #302, Nanaimo, BC V9R 0B8 Canada and (separately) by Sergio Falcón and Ángel Plaza (jointly), Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Las Palmas G. C., Spain.

The proposed identities appear in a slightly altered form in the following reference:

References

[1] R. S. Melham, *On certain combinations of high powers of Fibonacci numbers*, The Fibonacci Quarterly, **48.3** (2010), 256–259.

If we replace n by $n-1$ in Equations (1.4) and (1.8) of [1], we obtain the desired identities.

Solution II by Zbigniew Jakubczyk, Warsaw, Poland

We can easily prove the identity:

$$(a-b)^4 + 4a^4 + 4b^4 + (a+b)^4 = 6(a^2+b^2)^2.$$

Let $b = F_n$, $a = F_{n+1}$. Since

$$F_{n-1} = F_{n+1} - F_n = a - b$$
$$F_{n+2} = F_{n+1} + F_n = a + b$$
$$F_{2n+1} = F_n^2 + F_{n+1}^2 = a^2 + b^2,$$

we get identity (1). Identity (2) follows from the identity:

$$(a-b)^6 + 8a^6 + 8b^6 + (a+b)^6 = 10(a^2+b^2)^3$$

in a similar fashion.

Also solved by Charles K. Cook, Russell J. Hendel, George A. Hisert, Seung Hee Lee (student), Jaroslav Seibert, and the proposer.

Weighted Averages Type Inequality

B-1072 Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain (Vol. 48.3, August 2010)

Let n be a positive integer. For any real number, $\gamma > 1$, show that

$$\frac{1}{\gamma}\sum_{k=1}^{n}\left(F_k^{2\gamma}L_k^{2(1-\gamma)} + (\gamma-1)L_k^2\right) \ge F_nF_{n+1}.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan

Using Bernoulli's inequality, we have

$$\left(1+\left(\frac{F_k^2}{L_k^2}-1\right)\gamma\right)^{\frac{1}{\gamma}} \le \frac{F_k^2}{L_k^2}.$$

Therefore, we obtain the inequality

$$F_k^{2\gamma}L_k^{2(1-\gamma)} + (\gamma-1)L_k^2 - \gamma F_k^2 \ge 0.$$

Using this inequality and the identity $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$, we get

$$0 \le \sum_{k=1}^{n}(F_k^{2\gamma}L_k^{2(1-\gamma)} + (\gamma-1)L_k^2 - \gamma F_k^2) = \sum_{k=1}^{n}(F_k^{2\gamma}L_k^{2(1-\gamma)} + (\gamma-1)L_k^2) - \gamma F_nF_{n+1}.$$

The desired inequality follows.

Also solved by Paul S. Bruckman, Zbigniew Jakubczyk, Jaroslav Seibert, and the proposer.

A Diophantine Triple in Fibonacci Numbers

B-1073 Proposed by M. N. Deshpande, Nagpur, India (Vol. 48.3, August 2010)

Three integers (a, b, c) form a Diophantine Triple (DT) if and only if $ab+1$, $ac+1$, and $bc+1$ are perfect squares. It is known that $(F_{2n},\ F_{2n+2},\ F_{2n+4})$ is a DT for every integer n. If n is odd, prove that there exists an integer m such that $(m-F_{2n+4}, m-F_{2n+2}, m-F_{2n})$ is a DT. Also, if $n = 2k+1$ and the corresponding m is denoted by m_k, derive a recurrence relation involving m_k.

Solution by Brian P. Beasley, Department of Mathematics, Presbyterian College, Clinton, SC

For $k \ge 0$, we let $n = 2k+1$ and define $m_{k+2} = 7m_{k+1} - m_k$ with $m_0 = 8$ and $m_1 = 56$. Then $m_k = C\alpha^{4k} + D\beta^{4k}$, where $C = 4 + 28\sqrt{5}/15$ and $D = 4 - 28\sqrt{5}/15$. Let $x_k = \sqrt{(m_k - F_{2n+4})(m_k - F_{2n+2}) + 1}$, $y_k = \sqrt{(m_k - F_{2n+4})(m_k - F_{2n}) + 1}$, and $z_k = \sqrt{(m_k - F_{2n+2})(m_k - F_{2n}) + 1}$. Then

$$x_k^2 = \left(\frac{7}{18} + \frac{\sqrt{5}}{6}\right)\alpha^{8k} + \left(\frac{7}{18} - \frac{\sqrt{5}}{6}\right)\beta^{8k} + \frac{2}{9},$$

so $x_k = (1/2 + \sqrt{5}/6)\alpha^{4k} + (1/2 - \sqrt{5}/6)\beta^{4k}$. Similarly, we note that

$$y_k = \left(\frac{1}{2} + \frac{7\sqrt{5}}{30}\right)\alpha^{4k} + \left(\frac{1}{2} - \frac{7\sqrt{5}}{30}\right)\beta^{4k}$$

and

$$z_k = \left(3 + \frac{4\sqrt{5}}{3}\right)\alpha^{4k} + \left(3 - \frac{4\sqrt{5}}{3}\right)\beta^{4k}.$$

These sequences satisfy the following recurrence relations: $x_0 = 1$, $x_1 = 6$, $x_{k+2} = 7x_{k+1} - x_k$; $y_0 = 1$, $y_1 = 7$, $y_{k+2} = 7y_{k+1} - y_k$; $z_0 = 6$, $z_1 = 41$, $z_{k+2} = 7z_{k+1} - z_k$. (In fact, $m_k = 8y_k$ and $z_k = x_{k+1}$.) Hence, x_k, y_k, and z_k are integers for every k, so $(m_k - F_{2k+4}, m_k - F_{2n+2}, m_k - F_{2n})$ is a DT.

We note that when $k = 0$, the DT is (0, 5, 7); for $k > 0$, each element in the DT is positive. If negative integers are allowed in a DT, then $m = 0$ produces a trivial solution to the original problem. There is no recurrence involving m in this case.

Also solved by Paul S. Bruckman, Charles K. Cook, Russell J. Hendel, and the proposer.

A HM-GM Inequality Application

B-1074 Proposed by Pantelimon George Popescu, Bucureşt, România and José Luis Díaz-Barrero, Universidad Politécnica de Cataloñia, Barcelona, Spain (Vol. 48.3, August 2010)

Let $n \geq 3$ be a positive integer. Prove that

$$\frac{1}{\sqrt{1 - \frac{1}{F_n^2}}} + \frac{1}{\sqrt{1 - \frac{1}{L_n^2}}} > \frac{2}{\sqrt{1 - \left(\frac{F_{n+1}}{F_{2n}}\right)^2}}.$$

Solution by ONU-Solve problem group, Department of Mathematics and Statistics, ONU, Ada, OH

If $n \geq 3$, the harmonic-quadratic mean inequality gives

$$\frac{2}{\frac{1}{\sqrt{1-\frac{1}{F_n^2}}} + \frac{1}{\sqrt{1-\frac{1}{L_n^2}}}} < \sqrt{\frac{\left(\sqrt{1 - \frac{1}{F_n^2}}\right)^2 + \left(\sqrt{1 - \frac{1}{L_n^2}}\right)^2}{2}} = \sqrt{1 - \frac{1}{2}\left(\frac{1}{F_n^2} + \frac{1}{L_n^2}\right)}. \tag{1}$$

Note that the inequality in (1) is strict since $F_n \neq L_n$. Indeed, $L_n = F_{n+1} + F_{n-1} > F_n$ for $n \geq 2$. Next, we will show that

$$\frac{1}{2}\left(\frac{1}{F_n^2} + \frac{1}{L_n^2}\right) > \frac{F_{n+1}^2}{F_{2n}^2} \tag{2}$$

holds for $n \geq 2$. Indeed, if we use $F_{2n} = F_n L_n$, we see that (2) is equivalent to

$$F_n^2 + L_n^2 > 2F_{n+1}^2, \tag{3}$$

and if we further use the identity $2F_{n+1} = F_n + L_n$, we see that (3) is equivalent to

$$2F_n^2 + 2L_n^2 > (F_n + L_n)^2, \tag{4}$$

or $(F_n - L_n)^2 > 0$, which holds for all $n \geq 2$ since $F_n \neq L_n$. Finally, if $n \geq 3$, from (1) and (2) we obtain

$$\frac{2}{\frac{1}{\sqrt{1-\frac{1}{F_n^2}}} + \frac{1}{\sqrt{1-\frac{1}{L_n^2}}}} < \sqrt{1 - \left(\frac{F_{n+1}}{F_{2n}}\right)^2},$$

or

$$\frac{1}{\sqrt{1 - \frac{1}{F_n^2}}} + \frac{1}{\sqrt{1 - \frac{1}{L_n^2}}} > \frac{2}{\sqrt{1 - \left(\frac{F_{n+1}}{F_{2n}}\right)^2}}$$

which is the desired result.

Also solved by Paul S. Bruckman, Russell J. Hendel, Zbigniew Jakubczyk, and the proposer.

An "Inverse" Relation

B-1075 Proposed by Paul S. Bruckman, Nanaimo, BC, Canada (Vol. 48.3, August 2010)

The Fibonacci polynomials $F_n(x)$ may be defined by the following expression:

$$F_{n+1}(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k} \text{ for } n = 0, 1, 2, \ldots.$$

Prove the "inverse" relation:

$$x^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{n+1-2k}(x) \text{ for } n = 0, 1, 2, \ldots.$$

No solutions, other than the proposer's, were received for this problem. The deadline will be extended another three months.

A Closed Form For a Finite Product

B-1076 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 48.3, November 2010)

Find the closed form expression for

$$\prod_{k=1}^{n} (L_{2^{k+1}} - L_{2^k} + 1).$$

Solution by Paul S. Bruckman, Nanaimo, BC, Canada

Let P_n denote the indicated product, $n = 1, 2, \ldots$. Write $m = m(k) = 2^k$, for brevity. Then

$$\begin{aligned}
P_n &= \prod_{k=1}^{n} (L_{2m} - L_m + 1) = \prod_{k=1}^{n} (L_m^2 - L_m - 1) = \prod_{k=1}^{n} (L_m - \alpha)(L_m - \beta) \\
&= \prod_{k=1}^{n} (\alpha^m - \alpha + \alpha^{-m})(\alpha^m - \beta + \alpha^{-m}) \\
&= \prod_{k=1}^{n} (\alpha^{2m} - \alpha^m + 1 - \alpha^{-m} + \alpha^{-2m}) \\
&= \prod_{k=1}^{n} \left(\frac{\alpha^{5m/2} + \alpha^{-5m/2}}{\alpha^{m/2} + \alpha^{-m/2}} \right) = \left(\frac{\alpha^5 - \beta^5}{\alpha - \beta} \right) \prod_{k=1}^{n-1} \left(\frac{\alpha^{5m} + \beta^{5m}}{\alpha^m - \beta^m} \right) \\
&= 5 \prod_{k=1}^{n-1} \frac{L_{5m}}{L_m}.
\end{aligned}$$

Now,

$$F_2L_2L_4\dots L_{2^{n-1}} = F_4L_4\dots L_{2^{n-1}} = F_8\dots L_{2^{n-1}} = \dots = F_{2^n}.$$

Likewise,

$$F_{10}L_{10}L_{20}\dots L_{5\cdot 2^{n-1}} = F_{20}\dots L_{5\cdot 2^{n-1}} = F_{40}\dots L_{5\cdot 2^{n-1}} = \dots = F_{5\cdot 2^n}.$$

Then,

$$P_n = \frac{5F_2}{F_{10}}\cdot\frac{F_{5\cdot 2^n}}{F_{2^n}} = \frac{F_{5\cdot 2^n}}{11F_{2^n}}, n = 1, 2, \dots,$$

which is the desired closed form. Note that

$$P_1 = \frac{F_{10}}{11} = 5; P_2 = \frac{F_{20}}{11F_4} = \frac{55\cdot 123}{11\cdot 3} = 5\cdot 41 = 205;$$
$$P_3 = \frac{F_{40}}{11F_8} = \frac{55\cdot 123\cdot 15127}{11\cdot 3\cdot 7} = 5\cdot 41\cdot 2161 = 443005;$$

etc.

Also solved by the proposer.