ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2012. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

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BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1106 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Prove that

$$\sum_{k=1}^{3n} F_{2F_k} \equiv 0 \pmod{5}.$$

B-1107 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Determine

$$\sum_{j\geq 1, k\geq 3} \frac{1}{F_k^{4j-2}}.$$

B-1108 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania

Let $T_k = \frac{k(k+1)}{2}$ for all $k \geq 1$. Prove that

$$\sum_{k=1}^{n} \frac{F_k^{2m+2}}{T_k^m} \geq \frac{3^m (F_n F_{n+1})^{m+1}}{n^m T_{n+1}^m}$$

for any positive integer $n \geq 1$ and for any positive real number m.

B-1109 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania

Prove that

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4) > 9(F_n F_{n+1} F_{n+2})^{\frac{4}{3}}; \tag{1}$$

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 > \sqrt{6}(F_n F_{n+1} F_{n+2})^{\frac{2}{3}}; \tag{2}$$

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 \left(\frac{1}{F_n^4} + \frac{1}{F_{n+1}^4} + \frac{1}{F_{n+2}^4}\right) > 18; \tag{3}$$

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4) \left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right)^2 > 81. \tag{4}$$

B-1110 Proposed by Sergio Falcón and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

For any positive integer k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$, with respective initial conditions

$F_{k,0} = 0$; $F_{k,1} = 1$ and $L_{k,0} = 2$; $L_{k,1} = k$. Prove that

$$\sum_{i\ge 0}\binom{2n}{i}F_{k,i}^2 = (k^2+4)^{n-1}L_{k,2n} \tag{1}$$

$$\sum_{i\ge 0}\binom{2n+1}{i}F_{k,i-1}^2 = (k^2+4)^{n}F_{k,2n+1} \tag{2}$$

$$\sum_{i\ge 0}\binom{2n}{i}L_{k,i}^2 = (k^2+4)^{n}L_{k,2n} \tag{3}$$

$$\sum_{i\ge 0}\binom{2n+1}{i}L_{k,i}^2 = (k^2+4)^{n+1}F_{k,2n+1}. \tag{4}$$

SOLUTIONS

It Is Two!

B-1086 Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain (Vol. 49.2, May 2011)

Let n be a positive integer. Prove that

$$\frac{F_{2n+1}+F_nF_{n+1}+1}{F_{n+2}+\displaystyle\sum_{1\le i<j\le n}F_iF_j}$$

is an integer and determine its value.

Solution by Zbigniew Jakubczyk, Warsaw, Poland.

Using the well-known identities

$$\sum_{k=1}^{n}F_k = F_{n+2}-1,\ \sum_{k=1}^{n}F_k^2 = F_nF_{n+1},\ F_{2n+1} = F_n^2+F_{n+1}^2,\ \text{and } F_{n+2} = F_{n+1}+F_n,$$

we obtain

$$(F_{n+2}-1)^2 = \left(\sum_{k=1}^{n}F_k\right)^2 = \sum_{k=1}^{n}F_k^2 + 2\sum_{1\le i<j\le n}F_iF_j = F_nF_{n+1} + 2\sum_{1\le i<j\le n}F_iF_j.$$

Thus,

$$\frac{F_{2n+1}+F_nF_{n+1}+1}{F_{n+2}+\sum_{1\le i<j\le n}F_iF_j} = \frac{F_n^2+F_{n+1}^2+F_nF_{n+1}+1}{F_{n+2}+\frac{(F_{n+2}-1)^2-F_nF_{n+1}}{2}} = \frac{2(F_n^2+F_{n+1}^2+F_nF_{n+1}+1)}{F_{n+2}^2+1-F_nF_{n+1}}$$

$$= \frac{2(F_n^2+F_{n+1}^2+F_nF_{n+1}+1)}{F_{n+1}^2+F_n^2+F_nF_{n+1}+1} = 2.$$

Also solved by Paul S. Bruckman, Brian D. Beasley, M. N. Deshpande, Steve Edwards, Russell J. Hendel, Robinson Higuita, Harris Kwong, Carl Libis, Ángel Plaza, Jaroslav Seibert, James Sellers, and the proposer.

It Is Two, Too!

B-1087 Proposed by Br. J. Mahon, Kensington, Australia (Vol. 49.2, May 2011)

Evaluate

$$\prod_{i=3}^{\infty} \frac{F_{2i-1}^2 + F_{2i-1} - 2}{F_{2i-1}^2 - F_{2i-1} - 2}.$$

Solution by Paul S. Bruckman, Nanaimo, BC, Canada

We begin with the known relation:

$$F_{2n+2}F_{2n} - F_{2n+1}^2 = -1.$$

Then

$$\frac{(F_{2n+2}-1)(F_{2n}+1)}{(F_{2n}-1)(F_{2n+2}+1)} = \frac{F_{2n+1}^2 - 1 + F_{2n+2} - F_{2n} - 1}{F_{2n+1}^2 - 1 - F_{2n+2} + F_{2n} - 1} = \frac{F_{2n+1}^2 + F_{2n+1} - 2}{F_{2n+1}^2 - F_{2n+1} - 2}. \tag{1}$$

We now form the product of the expressions in (1), from $n = 2$ to $n = N - 1$, letting such product be denoted as P_N (supposing $N \geq 3$). Therefore, let

$$P_N = \prod_{n=2}^{N-1} \frac{(F_{2n+2}-1)(F_{2n}+1)}{(F_{2n}-1)(F_{2n+2}+1)} = \prod_{n=2}^{N-1} \left\{ \frac{F_{2n+1}^2 + F_{2n+1} - 2}{F_{2n+1}^2 - F_{2n+1} - 2} \right\} = \prod_{j=3}^{N} \left\{ \frac{F_{2j-1}^2 + F_{2j-1} - 2}{F_{2j-1}^2 - F_{2j-1} - 2} \right\}. \tag{2}$$

The first expression in (2) is a telescoping product; we easily see that

$$P_N = \frac{(F_{2N}-1)(F_4+1)}{(F_4-1)(F_{2N}+1)} = \frac{2(F_{2N}-1)}{F_{2N}+1}.$$

Then we see that $\lim_{N\to\infty}(P_N) = 2$. It follows that

$$\prod_{j=3}^{\infty} \left\{ \frac{F_{2j-1}^2 + F_{2j-1} - 2}{F_{2j-1}^2 - F_{2j-1} - 2} \right\} = 2.$$

Also solved by Kenneth B. Davenport, M. D. Deshpande, Robinson Higuita, and the proposer.

A "Well-Connected" Sequence

B-1088 Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain (Vol. 49.2, May 2011)

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers defined by $a_1 = 3$, $a_2 = 5$ and for all $n \geq 3$, $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$. Prove that

$$1 + \left(\sum_{k=1}^{n} \frac{L_k}{\sqrt{1+a_k}}\right)^2 < \frac{L_n L_{n+1}}{2}.$$

Solution by Ángel Plaza and Sergio Falcón (jointly), Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain

Since $\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2$, the proposed inequality is equivalent to

$$1 + \left(\sum_{k=1}^{n} \frac{L_k}{\sqrt{1+a_k}}\right)^2 < 1 + \frac{1}{2}\sum_{k=1}^{n} L_k^2$$

$$\left(\sum_{k=1}^{n} \frac{L_k}{\sqrt{1+a_k}}\right)^2 < \sum_{k=1}^{n} \frac{L_k^2}{2}.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{k=1}^{n} \frac{L_k}{\sqrt{1+a_k}}\right)^2 \leq \left(\sum_{k=1}^{n} L_k^2\right)\left(\sum_{k=1}^{n} \frac{1}{1+a_k}\right).$$

It is easy to check that the sequence $\{a_n\}_{n\geq 1}$ is related to the Silvester's sequence $\{b_n\}_{n\geq 1}$ which is defined by $b_{n+1} = b_n^2 - b_n + 1$, $b_1 = 2$, by the following relation,

$$a_n = 2b_n - 1.$$

Since $\sum_{k=1}^{\infty} \frac{1}{b_k} = 1$ (see [1]), the result $\sum_{k=1}^{n} \frac{1}{1+a_k} < \frac{1}{2}$ follows and the problem is done.

References

[1] A. V. Aho and N. J. A. Sloane, *Some doubly exponential sequences*, The Fibonacci Quarterly, **11.5** (1973), 429–437.

A similar proof was provided by K. B. Davenport who pointed out the connection between the sequence $\{a_k\}_{k\geq 1}$ and the "Greedy Odd" Egyptian fraction attributed to Erdös and Straus.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Robinson Higuita, Zbigniew Jakubczyk, Jaroslav Seibert, and the proposer.

A "Loaded Divisibility" Problem

B-1089 Proposed by Mohammad K. Azarian, University of Evansville, Indiana (Vol. 49.2, May 2011)

Let the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ be defined by the recurrence relation

$$\frac{a_{n+2}}{a_n} = a_{n+1}, n \geq 0, a_0 = 1, a_1 = 2.$$

Also, let $p_1, p_2, p_3, \ldots, p_n$, represent a permutation of $a_1, a_2, a_3, \ldots, a_n$. Show that

$$\prod_{j=1}^{n!}\left(\sum_{i=1}^{n}|a_i - p_i|\right) a_j$$

is divisible by $2^{n!+F_{n!+2}-1}$ for $n \geq 0$.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Notice that

$$\prod_{j=1}^{n!}\left(\sum_{i=1}^{n}|a_i - p_i|\right) a_j = \left(\sum_{i=1}^{n}|a_i - p_i|\right)^{n!} \cdot \prod_{j=1}^{n!} a_j.$$

It is easy to verify that $a_i = 2^{F_i}$. Hence, $\sum_{i=1}^{n}|a_i - p_i|$ is divisible by 2, and

$$\prod_{j=1}^{n!} a_j = 2^{\sum_{j=1}^{n!} F_j} = 2^{F_{n!+2}-1},$$

from which the desired result follows immediately.

Also solved by Robinson Higuita and Oscar Garcia (jointly), and the proposer.

A "Sterling" Sum

B-1090 Proposed by N. Gauthier, Kingston, ON, Canada (Vol. 49.2, May 2011)

a) Let x be an arbitrary variable and k, m, n, r nonnegative integers, with $0 \leq k \leq n$ and $0 \leq r \leq m$. Also let $\{S_r^{(m)} : 0 \leq r \leq m\}$ be the set of Stirling numbers of the second kind, which satisfy the following recurrence, with initial value $S_0^{(0)} = 1$ and boundary conditions $S_{-1}^{(m)} = 0$ and $S_{m+1}^{(m)} = 0$:

$$S_r^{(m+1)} = rS_r^{(m)} + S_{r-1}^{(m)}.$$

Finally, let $(n)_r = n(n-1)\cdots(n-r+1)$ for $r \neq 0$, with $(n)_0 = 1$, and adopt the convention that $k^0 = 1$ for all values of k, including zero. Prove that:

$$\sum_{k=0}^{n} k^m \binom{n}{k} x^k = \sum_{r=0}^{m} (n)_r S_r^{(m)} x^r (1+x)^{n-r}.$$

b) Use a) to show that

$$\sum_{k=0}^{n} k^4 \binom{n}{k} F_k = a(n)F_{2n-3} + b(n)F_{2n-4}$$

and determine the polynomial coefficients, $a(n)$ and $b(n)$.

Solution by Harris Kwong, SUNY Fredonia, NY

a) Induct on m. The identity holds when $m = 0$ because it reduces to the binomial theorem. Assume it holds for some integer $m \geq 0$, then

$$\begin{aligned}
\sum_{k=0}^{m} k^{m+1}\binom{n}{k}x^k &= x\frac{d}{dx}\left[\sum_{k=0}^{m} k^m\binom{n}{k}x^k\right] \\
&= x\frac{d}{dx}\left[\sum_{r=0}^{m}(n)_r S_r^{(m)} x^r(1+r)^{n-r}\right] \\
&= \sum_{r=1}^{n}(n)_r S_r^{(m)}\cdot rx^r(1+x)^{n-r} + \sum_{r=0}^{n-1}(n)_r S_r^{(m)}\cdot(n-r)x^{r+1}(1+x)^{n-r-1} \\
&= \sum_{r=1}^{n}(n)_r rS_r^{(m)}x^r(1+x)^{n-r} + \sum_{r=0}^{n-1}(n)_{r+1}S_r^{(m)}x^{r+1}(1+x)^{n-(r+1)} \\
&= \sum_{r=1}^{n}(n)_r rS_r^{(m)}x^r(1+x)^{n-r} + \sum_{r=1}^{n}(n)_r S_{r-1}^{(m)}x^r(1+x)^{n-r} \\
&= \sum_{r=1}^{n}(n)_r S_r^{(m+1)}x^r(1+x)^{n-r},
\end{aligned}$$

which completes the induction because $S_0^{(m)} = 0$ for all $m > 0$.

b) When x equals either α or β, we have $1+x = x^2$, hence, $x^r(1+x)^{n-r} = x^{2n-r}$. Therefore, Binet's formula leads to

$$\sum_{k=0}^{n} k^m\binom{n}{k}F_k = \sum_{r=0}^{m}(n)_r S_r^{(m)}F_{2n-r}.$$

In particular, since $S_0^{(4)} = 0$, $S_1^{(4)} = S_4^{(4)} = 1$, $S_2^{(4)} = 7$, and $S_3^{(4)} = 6$, we find

$$\begin{aligned}
\sum_{k=0}^{n} k^4\binom{n}{k}F_k &= (n)_1F_{2n-1} + 7(n)_2F_{2n-2} + 6(n)_3F_{2n-3} + (n)_4F_{2n-4} \\
&= (n)_1(2F_{2n-3} + F_{2n-4}) + 7(n)_2(F_{2n-3} + F_{2n-4}) + 6(n)_3F_{2n-3} + (n)_4F_{2n-4} \\
&= (6n^3 - 11n^2 + 7n)F_{2n-3} + (n^4 - 6n^3 + 18n^2 - 12n)F_{2n-4}.
\end{aligned}$$

E. H. M. Brietzke provided two proofs; one similar to the featured solution and another one combinatorial in nature.

Also solved by Paul S. Bruckman, Eduardo H. M. Brietzke, Kenneth B. Davenport, Robinson Higuito, and the proposer.