ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2021. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-408 Proposed by Lawrence Somer, Washington, D.C. (Vol. 17.3, October 1979)

Let $d \in \{2, 3, \dots\}$ and $G_n = F_{dn}/F_n$. Let p be an odd prime and $z = z(p)$ be the least positive integer n with $F_n \equiv 0 \pmod{p}$. For $d = 2$ and $z(p)$ an even integer $2k$, it was shown in B-386 that

$$F_{n+1}G_{n+k} \equiv F_n G_{n+k+1} \pmod{p}.$$

Establish a generalization for $d \geq 2$.

Editor's Note: This is another old problem from more than 40 years ago. No solutions have appeared, so we feature the problem again, and invite the readers to solve it.

B-1276 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{(L_n^2 - 5)^2} = \frac{1}{3}.$$

B-1277 **Proposed by Ivan V. Fedak, Precarpathian National University, Ivano-Frankivsk, Ukraine.**

For all positive integers n, prove that

$$\frac{F_{n-1}^2}{2F_{n+2}} \le \sqrt{\frac{F_{2n+1}}{2}} - \sqrt{\sum_{k=1}^{n} F_k^2} \le \frac{F_{n-1}^2}{F_{n+2}}.$$

B-1278 **Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.**

Show that the finite product

$$\prod_{k=0}^{n} \frac{F_{k+2}^2 + 2F_{k+1}F_{k+2}}{L_k F_{k+2} + (-1)^{k+1}}$$

is divisible by L_{n+2} for each integer $n \ge 0$.

B-1279 **Proposed by Pridon Davlianidze, Tbilisi, Republic of Georgia.**

Prove that

(A) $\displaystyle\prod_{n=1}^{\infty}\left(1 + \frac{1}{F_{2n}F_{2n+1}}\right) = \alpha,$

(B) $\displaystyle\prod_{n=1}^{\infty}\left(1 - \frac{1}{F_{2n-1}F_{2n+2}}\right) = \frac{1}{\alpha}.$

B-1280 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

The Tetranacci numbers T_n satisfy

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}, \qquad \text{for } n \ge 3,$$

with $T_{-1} = T_0 = 0$ and $T_1 = T_2 = 1$. Find a closed form expression for the sum $\displaystyle\sum_{k=1}^{n}(-1)^k T_k^2$.

SOLUTIONS

Another Oldie from the Vault

B-886 **Proposed by Peter J. Ferraro, Roselle Park, NJ.**
(Vol. 37.4, November 1999)

For $n \ge 9$, show that $\left\lfloor \sqrt[4]{F_n} \right\rfloor = \left\lfloor \sqrt[4]{F_{n-4}} + \sqrt[4]{F_{n-8}} \right\rfloor$.

Solution by Raphael Schumacher (student), ETH Zurich, Switzerland.

The result for $9 \le n \le 15$ can be verified by explicit computation, so we will assume $n \ge 16$.

Using the Catalan identity, we find $F_{n-4}F_{n-8} = F_{n-6}^2 + (-1)^{n+1}$. According to Taylor expansion, $\left|\sqrt{1+x}-1\right| \le |x|$ over $[-1,1]$. Thus,

$$\sqrt{F_{n-4}F_{n-8}} = F_{n-6}\sqrt{1+\frac{(-1)^{n+1}}{F_{n-6}^2}} = F_{n-6} + E_1(n),$$

with

$$|E_1(n)| \le \frac{1}{F_{n-6}} \le \frac{1}{55}.$$

We also find

$$\begin{aligned} F_{n-4}^3F_{n-8} &= F_{n-4}^2\left[F_{n-6}^2 + (-1)^{n+1}\right] \\ &= \big(F_{n-4}F_{n-6}\big)^2 + (-1)^{n+1}F_{n-4}^2 \\ &= \left[F_{n-5}^2 + (-1)^{n+1}\right]^2 + (-1)^{n+1}F_{n-4}^2 \\ &= F_{n-5}^4 + (-1)^{n+1}\big(F_{n-4}^2 + 2F_{n-5}^2\big) + 1. \end{aligned}$$

Because $\left|\sqrt[4]{1+x}-1\right| \le |x|$ over $[-1,1]$, we deduce that

$$\sqrt[4]{F_{n-4}^3F_{n-8}} = F_{n-5}\sqrt[4]{1+\frac{(-1)^{n+1}\big(F_{n-4}^2+2F_{n-5}^2\big)+1}{F_{n-5}^4}} = F_{n-5} + E_2(n),$$

where

$$|E_2(n)| \le \frac{F_{n-4}^2+2F_{n-5}^2+1}{F_{n-5}^3} \le \frac{5}{F_{n-5}} \le \frac{5}{89}.$$

In a similar manner, we also determine that

$$\sqrt[4]{F_{n-4}F_{n-8}^3} = F_{n-7}\sqrt[4]{1+\frac{(-1)^{n+1}\big(F_{n-8}^2+2F_{n-7}^2\big)+1}{F_{n-7}^4}} = F_{n-7} + E_3(n),$$

where

$$|E_3(n)| \le \frac{F_{n-8}^2+2F_{n-7}^2+1}{F_{n-7}^3} \le \frac{4}{F_{n-7}} \le \frac{2}{17}.$$

Therefore,

$$\begin{aligned} \left(\sqrt[4]{F_{n-4}} + \sqrt[4]{F_{n-8}}\right)^4 &= F_{n-4} + 4\sqrt[4]{F_{n-4}^3F_{n-8}} + 6\sqrt{F_{n-4}F_{n-8}} + 4\sqrt[4]{F_{n-4}F_{n-8}^3} + F_{n-8} \\ &= F_{n-4} + 4F_{n-5} + 6F_{n-6} + 4F_{n-7} + F_{n-8} + A(n) \\ &= F_n + A(n), \end{aligned}$$

where $A(n) = 4E_2(n) + 6E_1(n) + 4E_3(n)$, with $|A(n) < 1$. It is well known that $F_1 = F_2 = 1$ and $F_{12} = 144$ are the only square Fibonacci numbers [1]. This implies that $F_1 = F_2 = 1$ are the only Fibonacci numbers that are perfect fourth powers. Hence, there exists an integer m such that

$$m^4 < m^4 + 1 \le F_n \le (m+1)^4 - 1 < (m+1)^4.$$

Because $|A(n)| < 1$, we also have

$$m^4 < \left(\sqrt[4]{F_{n-4}} + \sqrt[4]{F_{n-8}}\right)^4 < (m+1)^4.$$

It follows immediately that

$$m = \left\lfloor \sqrt[4]{F_n} \right\rfloor = \left\lfloor \sqrt[4]{F_{n-4}} + \sqrt[4]{F_{n-8}} \right\rfloor$$

for $n \geq 16$.

Editor's Note: Plaza noted that a more general problem appeared in [2]. The partial solution that appeared in Vol. 108 (2001), 978–979, of the same journal yields the desired result as a special case.

References

[1] J. H. E. Cohn, *On square Fibonacci numbers*, J. London Math. Soc., **39** (1964), 537–540.
[2] Peter J. Ferraro, *Problem 10765*, Amer. Math. Monthly, **106** (1999), 864.

Also solved by G. C. Greubel, Ángel Plaza, Albert Stadler, and the proposer.

Solving a Quadratic Equation

B-1256 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine. (Vol. 57.4, November 2019)

For any positive integers n, find an infinite set of pairs of positive Fibonacci numbers x and y such that

$$x^2 - xy - y^2 = F_n F_{n+1} - F_{n-1} F_{n+2}.$$

Solution by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

First, we note that

$$F_n F_{n+1} - F_{n-1} F_{n+2} = (-1)^{n+1}. \tag{1}$$

Hence, we seek to find solutions of the equation

$$y^2 + xy - x^2 + (-1)^{n+1} = 0,$$

where n is a *fixed* positive integer. Solving for y, we find

$$y = \frac{-x \pm \sqrt{5x^2 + 4(-1)^n}}{2}.$$

From the identity $L_t^2 = 5F_t^2 + 4(-1)^t$, we see that we can choose $x = F_{mn+k}$, provided $(-1)^{mn+k} = (-1)^n$. Hence, we also need m odd and k even. Using $L_t = F_{t+1} + F_{t-1}$, we are able to simplify the value of y. We obtain an infinite set of pairs of solutions

$$(x, y) = (F_{mn+k}, F_{mn+k-1}), (F_{mn+k}, -F_{mn+k+1}), \qquad m \text{ odd and } k \text{ even}.$$

A closing remark: the automorphism $(x, y) \mapsto (2x + y, x + y)$ produces additional solutions, which in our case have the same form.

Editor's Note: A number of solvers proposed, for example, $(x, y) = (F_n, F_{n-1})$ as a solution. However, because n is fixed, this in effect provides only *one* solution. Davlianidze remarked that $5x^2 \pm 4$ is a perfect square if and only if x is a Fibonacci number [1]. It is easy to verify that the generalized Fibonacci numbers defined by $G_n = G_{n-1} + G_{n-2}$ also satisfy (1), it follows that, as the proposer noted, we can replace F_t with G_t in the solution.

Reference

[1] I. Gessel, *Solution to Problem H-187*, The Fibonacci Quarterly, **10.4** (1972), 417–419.

Also solved by Michel Bataille, Brian D. Beasley, Brian Bradie, Pridon Davlianidze, G. C. Greubel, Ángel Plaza, Raphael Schumacher (student), David Terr, and the proposer.

Make It Telescope

B-1257 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.
(Vol. 57.4, November 2019)

Find closed form expressions for the alternating sums

$$\sum_{k=0}^{n}(-1)^k F_{3^k}F_{2\cdot 3^k} \qquad \text{and} \qquad \sum_{k=0}^{n}(-1)^k F_{3^k}L_{2\cdot 3^k}.$$

Solution by Dmitry Fleischman, Santa Monica, CA.

Because $\alpha\beta = -1$ and $(-1)^{3^k} = -1$, we deduce from the Binet's formula that

$$F_{3^k}F_{2\cdot 3^k} = \frac{\left(\alpha^{3^k} - \beta^{3^k}\right)\left(\alpha^{2\cdot 3^k} - \beta^{2\cdot 3^k}\right)}{5} = \frac{\alpha^{3^k} + \beta^{3^k} + \alpha^{3^{k+1}} + \beta^{3^{k+1}}}{5} = \frac{1}{5}\left(L_{3^k} + L_{3^{k+1}}\right).$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k F_{3^k}F_{2\cdot 3^k} &= \frac{1}{5}\left[L_1 + L_3 - L_3 - L_9 + L_9 + L_{27} - \cdots + (-1)^n\left(L_{3^n} + L_{3^{n+1}}\right)\right]\\ &= \frac{1}{5}\left[1 + (-1)^n L_{3^{n+1}}\right].\end{aligned}$$

Similarly, from

$$F_{3^k}L_{2\cdot 3^k} = \frac{\left(\alpha^{3^k} - \beta^{3^k}\right)\left(\alpha^{2\cdot 3^k} + \beta^{2\cdot 3^k}\right)}{\sqrt{5}} = \frac{\alpha^{3^k} - \beta^{3^k} + \alpha^{3^{k+1}} - \beta^{3^{k+1}}}{\sqrt{5}} = F_{3^k} + F_{3^{k+1}},$$

we gather that

$$\sum_{k=0}^{n}(-1)^k F_{3^k}L_{2\cdot 3^k} = 1 + (-1)^n F_{3^{n+1}}.$$

Editor's Note: Edwards and Weiner (independently) used induction to derive the different but equivalent closed forms $(-1)^n F_{\frac{3^{n+1}-1}{2}} F_{\frac{3^{n+1}+1}{2}}$ and $(-1)^n F_{\frac{3^{n+1}-1}{2}} L_{\frac{3^{n+1}+1}{2}}$, respectively, for the two sums.

Also solved by Michel Bataille, Brian Bradie, Alejandro Cardona Castrillón (student), Steve Edwards, I. V. Fedak, G. C. Greubel, Hideyuki Ohtsuka, Raphael Schumacher (student), Albert Stadler, Dan Weiner, and the proposer.

Another Trigonometric Inequality

B-1258 Proposed by D. M. Bătineţu-Giurgiu, Mateo Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 57.4, November 2019)

Prove that

(i) $\sin\left(F_{2n+3}\right) + \sin\left(F_{n+1}F_n\right) + \cos\left(F_{n+3}F_{n+2}\right) \le \frac{3}{2}$

(ii) $\sin\left(F_m L_n\right) + \sin\left(F_n L_m\right) + \cos\left(2F_{m+n}\right) \le \frac{3}{2}$

Solution by Daniel Văcaru, Piteşti, Romania.

We first prove a lemma: for $A + B + C = \pi$, we have

$$\cos A + \cos B + \cos C \le \frac{3}{2}.$$

To prove the lemma, it suffices to prove that

$$\begin{aligned} \cos A + \cos B + \cos C - 1 &= 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} - 2\sin^2\frac{C}{2} \\ &= 2\sin\frac{C}{2}\sin\frac{A-B}{2} - 2\sin^2\frac{C}{2} \\ &\le \frac{1}{2}. \end{aligned}$$

Because $4\sin^2\frac{A-B}{2} - 4 \le 0$, we note that

$$-2t^2 + 2t\sin\frac{A-B}{2} - \frac{1}{2} \le 0$$

for all real numbers t. The lemma follows by setting $t = \sin\frac{C}{2}$.

From the shifting property $F_{s+t} = F_s F_{t+1} + F_{s-1}F_t$, we obtain

$$\begin{aligned} F_{n+3}F_{n+2} - F_{n+1}F_n &= (F_{n+2} + F_{n+1})F_{n+2} - F_{n+1}(F_{n+2} - F_{n+1}) \\ &= F_{n+2}^2 + F_{n+1}^2 \\ &= F_{2n+3}. \end{aligned}$$

Therefore,

$$\left(\frac{\pi}{2} - F_{2n+3}\right) + \left(\frac{\pi}{2} - F_{n+1}F_n\right) + F_{n+3}F_{n+2} = \pi.$$

Using the addition formula $F_{m+n} = \frac{1}{2}\left(F_m L_n + F_n L_m\right)$, we obtain

$$\left(\frac{\pi}{2} - F_m L_n\right) + \left(\frac{\pi}{2} - F_n L_m\right) + 2F_{m+n} = \pi.$$

The lemma immediately yields (i) and (ii).

Editor's Note: This problem is similar to Problem B-1253.

Also solved by Michel Baitaille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Hideyuki Ohtsuka, Albert Stadler, and the proposer.

Jensen's Inequality on a Convex Function

B-1259 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 57.4, November 2019)

Let k be a positive integer. The k-Fibonacci numbers are defined by the recurrence relation $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$, with initial values $F_{k,0} = 0$ and $F_{k,1} = 1$. Prove that

(i) $\displaystyle\sum_{i=1}^{n} \frac{F_{k,i}^2}{\sqrt{F_{k,i}+1}} \geq \frac{(F_{k,n}+F_{k,n+1}-1)^2}{k\sqrt{kn(F_{k,n}+F_{k,n+1}-1+kn)}}$

(ii) $\displaystyle\sum_{i=1}^{n} \frac{F_{k,i}^4}{\sqrt{F_{k,i}^2+1}} \geq \frac{F_{k,n}^2F_{k,n+1}^2}{k\sqrt{kn(F_{k,n}F_{k,n+1}+kn)}}$

Solution by Albert Stadler, Herrliberg, Switzerland.

We note that the function $f(x) = \frac{x^2}{\sqrt{x+1}}$ is convex over $\mathbb{R}^+$, because

$$f''(x) = \frac{3x^2+8x+8}{4(x+1)^{5/2}} > 0.$$

Hence, by Jensen's inequality,

$$\sum_{i=1}^{n} \frac{F_{k,i}^2}{\sqrt{F_{k,i}+1}} \geq \frac{n\left(\frac{1}{n}\sum_{i=1}^{n} F_{k,i}\right)^2}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} F_{k,i}+1}},$$

and

$$\sum_{i=1}^{n} \frac{F_{k,i}^4}{\sqrt{F_{k,i}^2+1}} \geq \frac{n\left(\frac{1}{n}\sum_{i=1}^{n} F_{k,i}^2\right)^2}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} F_{k,i}^2+1}}.$$

It remains to prove that

$$S := \sum_{i=1}^{n} F_{k,i} = \frac{1}{k}\left(F_{k,n+1}+F_{k,n}-1\right),$$

and

$$T := \sum_{i=1}^{n} F_{k,i}^2 = \frac{1}{k}F_{k,n}F_{k,n+1}.$$

To complete the proof, note that

$$0 = \sum_{i=1}^{n}\left(F_{k,i+1}-kF_{k,i}-F_{k,i-1}\right) = \left(S+F_{k,n+1}-1\right) - kS - \left(S-F_{k,n}\right),$$

and

$$0 = \sum_{i=0}^{n} F_{k,i}\left(F_{k,i+1}-kF_{k,i}-F_{k,i-1}\right) = F_{k,n}F_{k,n+1} - kT.$$

Also solved by Michel Bataille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, G. C. Greubel, and the proposer.

From Floor to Fibonacci Number

**B-1260 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 57.4, November 2019)**

For any positive integer n, find a closed form expression for the sum

$$\sum_{k=1}^{n} \left\lfloor \frac{F_k}{\alpha F_k - F_{k+1}} \right\rfloor .$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Starting from the Binet form for the Fibonacci numbers,

$$\alpha F_k - F_{k+1} = \frac{\alpha^{k+1} + \beta^{k-1}}{\sqrt{5}} - \frac{\alpha^{k+1} - \beta^{k+1}}{\sqrt{5}} = \frac{\beta^{k-1}(-\alpha\beta + \beta^2)}{\sqrt{5}} = -\beta^k.$$

Thus,

$$\frac{F_k}{\alpha F_k - F_{k+1}} = \frac{\alpha^k - \beta^k}{-\sqrt{5}\,\beta^k} = \frac{(-1)^{k+1}\alpha^{2k} + 1}{\sqrt{5}}.$$

Now, for k odd,

$$F_{2k} = \frac{\alpha^{2k} - \beta^{2k}}{\sqrt{5}} < \frac{\alpha^{2k} + 1}{\sqrt{5}} < \frac{\alpha^{2k} + \sqrt{5} - \beta^{2k}}{\sqrt{5}} = F_{2k} + 1,$$

while for k even,

$$-F_{2k} = \frac{-\alpha^{2k} + \beta^{2k}}{\sqrt{5}} < \frac{-\alpha^{2k} + 1}{\sqrt{5}} < \frac{-\alpha^{2k} + \sqrt{5} + \beta^{2k}}{\sqrt{5}} = -F_{2k} + 1;$$

therefore,

$$\left\lfloor \frac{F_k}{\alpha F_k - F_{k+1}} \right\rfloor = (-1)^{k+1} F_{2k}.$$

Using the double argument formula $F_{2n} = F_n L_n$ and the conjugation relation $L_n = F_{n-1} + F_{n+1}$,

$$F_{2k} = F_k L_k = F_k(F_{k-1} + F_{k+1}) = F_{k-1}F_k + F_k F_{k+1}.$$

Finally,

$$\sum_{k=1}^{n} \left\lfloor \frac{F_k}{\alpha F_k - F_{k+1}} \right\rfloor = \sum_{k=1}^{n} (-1)^{k+1}(F_{k-1}F_k + F_k F_{k+1}) = (-1)^{n+1} F_n F_{n+1}.$$

Also solved by Michel Bataille, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Ángel Plaza, Albert Stadler, David Terr, and the proposer.