ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2024. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1346 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Show that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{L_{2i}} \leq \frac{L_{2n}-2}{L_{2n+1}-1}.$$

Deduce that

$$\sum_{i=0}^{\infty} \frac{1}{L_{2i}} \leq \frac{\sqrt{5}}{2}.$$

B-1347 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For integers $m > n \geq 0$, prove that

$$\sum_{k=0}^{n} \binom{m}{k} (-1)^{n+k} F_{m-2k} = \sum_{k=n}^{m-1} \binom{k}{n} F_{k-2n-1}.$$

B-1348 **Proposed by Juan Pla, Paris, France.**

Prove that

(a) neither L_n nor F_n are multiples of 7 whenever n is odd, and
(b) L_n is a multiple of 7 if and only if $n = 8m + 4$ for some integer m.

B-1349 **Proposed by Toyesh Prakash Sharma (undergraduate), Agra College, Agra, India.**

For any integer $n \geq 2$, show that

$$F_n^{L_n} F_{n+1}^{F_{n+1}} L_n^{F_n} \leq F_{2n}^{F_{n+1}}.$$

B-1350 **Proposed by Michel Bataille, Rouen, France.**

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{F_{3n}} \leq \frac{4}{9} \sum_{n=1}^{\infty} \frac{(-1)^{1+F_n}}{F_n}.$$

SOLUTIONS

The Powers of i

B-1326 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**
(Vol. 61.2, May 2023)

Let $i = \sqrt{-1}$. For any integer $n \geq 1$, prove that

$$\left| \sum_{k=1}^{n} i^k F_k \right| = F_{\lceil n/2 \rceil} \sqrt{F_{2\lfloor n/2 \rfloor + 1}}.$$

Solution by Steve Edwards, Roswell, GA.

First note that it is easy to show by induction that

$$\sum_{k=1}^{m} (-1)^{k+1} F_{2k-1} = (-1)^{m+1} F_m^2, \qquad \text{and} \qquad \sum_{k=1}^{m} (-1)^{k+1} F_{2k} = (-1)^{m+1} F_m F_{m+1}.$$

Using properties of the powers of i and these identities, we have

$$\begin{aligned} \sum_{k=1}^{n} i^k F_k &= i \sum_{k=1}^{\lceil n/2 \rceil} (-1)^{k+1} F_{2k-1} - \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k+1} F_{2k} \\ &= i\,(-1)^{\lceil n/2 \rceil + 1} F_{\lceil n/2 \rceil}^2 - (-1)^{\lfloor n/2 \rfloor + 1} F_{\lfloor n/2 \rfloor} F_{\lfloor n/2 \rfloor + 1}. \end{aligned}$$

Now for n even, $\lceil n/2 \rceil = n/2 = \lfloor n/2 \rfloor$, from which it follows that

$$\left| \sum_{k=1}^{n} i^k F_k \right| = F_{\lceil n/2 \rceil} \sqrt{F_{\lfloor n/2 \rfloor}^2 + F_{\lfloor n/2 \rfloor + 1}^2} = F_{\lceil n/2 \rceil} \sqrt{F_{2\lfloor n/2 \rfloor + 1}},$$

where we have used the identity $F_m^2 + F_{m+1}^2 = F_{2m+1}$.

For n odd, $\lceil n/2\rceil = n/2 + 1/2 = \lfloor n/2\rfloor + 1$, so

$$\left|\sum_{k=1}^{n} i^k F_k\right| = F_{\lceil n/2\rceil}\sqrt{F_{\lceil n/2\rceil}^2 + F_{\lfloor n/2\rfloor}^2} = F_{\lceil n/2\rceil}\sqrt{F_{\lfloor n/2\rfloor+1}^2 + F_{\lfloor n/2\rfloor}^2} = F_{\lceil n/2\rceil}\sqrt{F_{2\lfloor n/2\rfloor+1}}.$$

Editor's Note: Greubel found a Lucas analog:

$$\left|\sum_{k=0}^{n} i^k L_k\right| = \begin{cases} L_{\lceil n/2\rceil}\sqrt{F_{2\lfloor n/2\rfloor+1}} & n \text{ even},\\ F_{\lceil (n+1)/2\rceil}\sqrt{5F_{2\lfloor n/2\rfloor+1}} & n \text{ odd}.\end{cases}$$

Also solved by Thomas Achammer, Michael R. Bacon and Charles K. Cook (jointly), Michel Bataille, Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Richard G. Gonzalez Hernandez and Edwin Daniel Patiño Osorio (undergraduates) (jointly), G. C. Greubel, Ángel Plaza, Patrick Rappa, Raphael Schumacher (graduate student), David Terr, Eli Torek (undergraduate), Yunyong Zhang, and the proposer.

Two Infinite Series of Summations of Fibonacci/Lucas Numbers

B-1327 Proposed by Brian Bradie, Christopher Newport University, Newport News, VA.
(Vol. 61.2, May 2023)

For each nonnegative integer n, define

$$a_n = \left(\sum_{k=0}^{n} F_k\right)^2 - 2\sum_{k=0}^{n} F_k^2, \qquad \text{and} \qquad b_n = \left(\sum_{k=0}^{n} L_k\right)^2 - 2\sum_{k=0}^{n} L_k^2.$$

Evaluate $\displaystyle\sum_{n=0}^{\infty}\frac{a_n}{3^n}$ and $\displaystyle\sum_{n=0}^{\infty}\frac{b_n}{3^n}$.

Solution by Jason L. Smith, Richland Community College, Decatur, IL.

We shall extend the results to the generalized Fibonacci numbers defined as $G_1 = a$, $G_2 = b$, and

$$G_n = G_{n-1} + G_{n-2} \qquad \text{for } n \geq 3.$$

We want to evaluate $\sum_{n=0}^{\infty} c_n/3^n$, where

$$c_n = \left(\sum_{k=0}^{n} G_k\right)^2 - 2\sum_{k=0}^{n} G_k^2.$$

Using Problems 11 and 14 [1, p. 113], with $G_0 = b - a$, we find

$$\sum_{k=0}^{n} G_k = G_{n+2} - a, \qquad \text{and} \qquad \sum_{k=0}^{n} G_k^2 = G_n G_{n+1} + (b-a)(b-2a).$$

It is known [1, p. 109] that

$$G_n = aF_{n-2} + bF_{n-1},$$

which can be easily proved by induction. Therefore,

$$\begin{aligned} c_n &= (G_{n+2}-a)^2 - 2\left[G_nG_{n+1} + (b-a)(b-2a)\right] \\ &= a^2(F_n^2 - 2F_{n-2}F_{n-1}) + 2ab\,(F_nF_{n+1} - F_{n-1}^2 - F_{n-2}F_n) \\ &\qquad + b^2(F_{n+1}^2 - 2F_{n-1}F_n) - 2a^2F_n - 2abF_{n+1} - (3a^2 - 6ab + 2b^2). \end{aligned}$$

Since

$$F_n^2 - 2F_{n-2}F_{n-1} = (F_{n-1}+F_{n-2})^2 - 2F_{n-2}F_{n-1} = F_{n-1}^2 + F_{n-2}^2 = F_{2n-3},$$

we also obtain $F_{n+1}^2 - 2F_{n-1}F_n = F_{2n-1}$. In addition,

$$\begin{aligned} F_nF_{n+1} - F_{n-1}^2 - F_{n-2}F_n &= F_n(F_{n+1} - F_{n-2}) - F_{n-1}^2 \\ &= 2F_nF_{n-1} - F_{n-1}^2 = F_{n-1}(2F_n - F_{n-1}) = F_{n-1}L_{n-1} = F_{2n-2}. \end{aligned}$$

They lead to

$$c_n = a^2F_{2n-3} + 2abF_{2n-2} + b^2F_{2n-1} - 2a^2F_n - 2abF_{n+1} - (3a^2 - 6ab + 2b^2).$$

Next, we use generating functions to evaluate $\sum_{n=0}^{\infty} c_n/3^n$.

We have [1, p. 230]

$$\sum_{n=0}^{\infty} F_{2n+1}x^n = \frac{1-x}{1-3x+x^2}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} F_{2n}x^n = \frac{x}{1-3x+x^2}.$$

It follows that (since $F_{-3} = 2$ and $F_{-1} = 1$)

$$\sum_{n=0}^{\infty} F_{2n-3}x^n = F_{-3} + F_{-1}x + x^2 \cdot \frac{1-x}{1-3x+x^2} = \frac{2-5x}{1-3x+x^2}.$$

In a similar fashion, we find (using $F_{-2} = -1$)

$$\sum_{n=0}^{\infty} F_{2n-2}x^n = \frac{-1+3x}{1-3x+x^2}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} F_{2n-1}x^n = \frac{1-2x}{1-3x+x^2}.$$

Hence,

$$\sum_{n=0}^{\infty} \frac{F_{2n-3}}{3^n} = 3, \qquad \sum_{n=0}^{\infty} \frac{F_{2n-2}}{3^n} = 0, \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{F_{2n-1}}{3^n} = 3.$$

From

$$\sum_{n=0}^{\infty} F_nx^n = \frac{x}{1-x-x^2}, \qquad \sum_{n=0}^{\infty} F_{n+1}x^n = \frac{1}{1-x-x^2}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

we deduce that

$$\sum_{n=0}^{\infty} \frac{F_n}{3^n} = \frac{3}{5}, \qquad \sum_{n=0}^{\infty} \frac{F_{n+1}}{3^n} = \frac{9}{5}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{3}{2}.$$

In conclusion, we find

$$\sum_{n=0}^{\infty} \frac{c_n}{3^n} = 3a^2 + 3b^2 - \frac{6a^2}{5} - \frac{18ab}{5} - \frac{3(3a^2-6ab+2b^2)}{2} = \frac{27a(2b-a)}{10}.$$

Since $c_n = a_n$ when $a = b = 1$, and $c_n = b_n$ when $a = 1$ and $b = 3$, we gather that

$$\sum_{n=0}^{\infty} \frac{a_n}{3^n} = \frac{27}{10}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{b_n}{3^n} = \frac{27}{2}.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons Inc., New York, NY, 2001.

Also solved by Thomas Achammer, Michel Bataille, Charles K. Cook and Michael R. Bacon (jointly), Kenny B. Davenport, Steve Edwards, I. V. Fedak. Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Edwin Daniel Patiño Osorio (undergraduate), Raphael Schumacher (graduate student), Albert Stadler, David Terr, Yunyong Zhang, and the proposer.

An Almost Trivial Inequality

B-1328 Proposed by Toyesh Prakash Sharma (undergraduate), Agra College, Agra, India.
(Vol. 61.2, May 2023)

For any integer $n \geq 0$, show that

$$\frac{2^{n+1}F_{2n+1}}{2n+1} \geq L_n.$$

Solution 1 by Brian D. Beasley, Simpsonville, SC.

The claim holds with equality for $n = 0$. For $n \geq 1$, we note that $2^{n+1} > 2n + 1$ and thus

$$\frac{2^{n+1}F_{2n+1}}{2n+1} > F_{2n+1} > F_{2n} = F_n L_n \geq L_n.$$

Solution 2 by Hideyuki Ohtsuka, Saitama, Japan.

For $n = 0$, the left and right sides equal 2. For $n \geq 1$, the inequality holds since

$$2^{n+1}F_{2n+1} > 2F_{2n} \sum_{k=0}^{n} \binom{n}{k} \geq 2F_n L_n(1 + n) > (2n + 1)L_n.$$

Also solved by Thomas Achammer, Michel Bataille, Brian D. Beasley (second solution), Brian Bradie, Kenny B. Davenport (two solutions), I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Ralph P. Grimaldi, Edwin Daniel Patiño Osorio (undergraduate), Ángel Plaza, Albert Stadler, Eli Torek (undergraduate), Andrés Ventas, and the proposer.

It Is Always Greater Than e^2

B-1329 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 61.2, May 2023)

For any positive integer n, prove that

$$\left(\frac{F_{6n}L_{2n}}{L_{6n}F_{2n}}\right)^{L_{4n}} > e^2.$$

Solution by I. V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

First, we note that

$$\frac{F_{6n}L_{2n}}{L_{6n}F_{2n}} = \frac{\alpha^{6n}-\beta^{6n}}{\alpha^{2n}-\beta^{2n}} \cdot \frac{\alpha^{2n}+\beta^{2n}}{\alpha^{6n}+\beta^{2n}} = \frac{\alpha^{4n}+\alpha^{2n}\beta^{2n}+\beta^{4n}}{\alpha^{4n}-\alpha^{2n}\beta^{2n}+\beta^{4n}} = \frac{L_{4n}+1}{L_{4n}-1}.$$

Next, consider the function $f(x) = \left(\frac{x+1}{x-1}\right)^x$, where $x > 1$. We have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}}\right)^{\frac{2x}{x-1}} = e^2.$$

Let $g(x) = \ln f(x) = x \ln\left(\frac{x+1}{x-1}\right)$. Then

$$g'(x) = \ln\left(\frac{x+1}{x-1}\right) - \frac{2x}{x^2-1}.$$

We see that $\lim_{x\to\infty} g'(x) = 0$. From here, since $g''(x) = \frac{4}{(x^2-1)^2} > 0$, we obtain $g'(x) < 0$ for all $x > 1$. Therefore, over the interval $(1,\infty)$, both functions $g(x)$ and $f(x)$ are decreasing. Thus, using $\lim_{x\to\infty} f(x) = e^2$, we deduce that $f(x) > e^x$ for all $x > 1$. In particular, if $x = L_{4n}$, where n is any positive integer,

$$\left(\frac{F_{6n}L_{2n}}{L_{6n}F_{2n}}\right)^{L_{4n}} = \left(\frac{L_{4n}+1}{L_{4n}-1}\right)^{L_{4n}} > e^2.$$

Also solved by Thomas Achammer, Michel Bataille, Brian D. Beasley, Brian Bradie, Kenny B. Davenport, Dmitry Fleischman, Won Kyun Jeong, Ángel Plaza, Albert Stadler, David Terr, Andrés Ventas, and the proposer.

Two Infinite Products Related to Fibonacci/Lucas Numbers

B-1330 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany. (Vol. 61.2, May 2023)

Two sequences of numbers x_n and y_n are defined by the same building rule $z_{n+1} = \frac{1+z_n}{4-z_n}$, $n \ge 0$, but with different initial values $x_0 = 0$ and $y_0 = 1$. Prove that

$$\prod_{k=1}^{n} x_k = \frac{1}{F_{2\lfloor n/2\rfloor+2}L_{2\lfloor (n-1)/2\rfloor+3}}, \qquad \text{and} \qquad \prod_{k=1}^{n} y_k = \frac{2}{F_{2\lfloor n/2\rfloor+1}L_{2\lfloor (n-1)/2\rfloor+2}}.$$

Solution by Michel Bataille, Rouen, France.

We will use the following remark: if $0 \le z_n \le 1$, then $z_{n+1} \neq 4$, and

$$z_{n+2} = \frac{1+z_{n+1}}{4-z_{n+1}} = \frac{1+\frac{1+z_n}{4-z_n}}{4-\frac{1+z_n}{4-z_n}} = \frac{1}{3-z_n}.$$

First, we show that for any $n \ge 1$,

$$x_{2n-1} = \frac{L_{2n-1}}{L_{2n+1}}, \quad x_{2n} = \frac{F_{2n}}{F_{2n+2}}, \quad y_{2n-1} = \frac{L_{2n-2}}{L_{2n}}, \quad y_{2n} = \frac{F_{2n-1}}{F_{2n+1}}. \tag{1}$$

Direct computation yield $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{3}$, $y_1 = \frac{2}{3}$, and $y_2 = \frac{1}{2}$, The relations in (1) hold for $n = 1$. Assume that (1) holds for some positive integer n. Since x_{2n-1}, x_{2n}, y_{2n-1}, y_{2n} are in the interval $[0, 1]$, the remark above gives

$$x_{2n+1} = \frac{1}{3 - x_{2n-1}} = \frac{1}{3 - \frac{L_{2n-1}}{L_{2n+1}}} = \frac{L_{2n+1}}{3L_{2n+1} - L_{2n-1}} = \frac{L_{2n+1}}{L_{2n+3}},$$

the latter because

$$3L_{2n+1} - L_{2n-1} = 2L_{2n+1} + L_{2n} = L_{2n+1} + L_{2n+2} = L_{2n+3}.$$

Similarly, we obtain

$$x_{2n+2} = \frac{1}{3 - x_{2n}} = \frac{1}{3 - \frac{F_{2n}}{F_{2n+2}}} = \frac{F_{2n+2}}{3F_{2n+2} - F_{2n}} = \frac{F_{2n+2}}{F_{2n+4}},$$

and

$$y_{2n+1} = \frac{L_{2n}}{3L_{2n} - L_{2n-2}} = \frac{L_{2n}}{L_{2n+2}}, \qquad y_{2n+2} = \frac{F_{2n+1}}{3F_{2n+1} - F_{2n-1}} = \frac{F_{2n+1}}{F_{2n+3}}.$$

This completes the induction showing that (1) holds for all $n \geq 1$.

From (1), we obtain that for any $m \geq 1$,

$$\prod_{k=1}^{2m} x_k = \frac{L_1 F_2}{L_{2m+1} F_{2m+2}} = \frac{1}{L_{2m+1} F_{2m+2}}, \qquad \prod_{k=1}^{2m} y_k = \frac{L_0 F_1}{L_{2m} F_{2m+1}} = \frac{2}{L_{2m} F_{2m+1}},$$

and for any $m \geq 0$

$$\prod_{k=1}^{2m+1} x_k = \frac{1}{F_{2m+2} L_{2m+3}}, \qquad \prod_{k=1}^{2m+1} y_k = \frac{2}{F_{2m+1} L_{2m+2}}.$$

They agree with the assertions in the problem statement.

Also solved by Thomas Achammer, Michael R. Bacon and Charles K. Cook (jointly), Molly George, Jade Melanson, Ty Miller and James Sun (high school students attending the 2023 Math Research Experience Program at the Citadel, Charleston, SC) (jointly). Steve Edwards, I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Edwin Daniel Patiño Osorio (undergraduate), Ángel Plaza, Raphael Schumacher (graduate student), Yunyong Zhang, and the proposer.

Editor's Note: Due to an oversight, historical comments by the proposer (Hans J. H. Tuenter) were inadvertently omitted when the solution to Problem B-1325 was published.

Tuenter's Comments and Historical References to Problem B-1325: We note that a similar problem was posed in the March 1969 issue of the *Scientific American*. In his column *Mathematical Games*, Martin Gardner [2] asked to show that the sum of 10 consecutive numbers in a generalized Fibonacci sequence is 11 times the seventh number. The problem can also be found in Gardner's book *Mathematics, Magic and Mystery* [1, pp. 158–159], where it is mentioned that its roots go back to 1940 as a performance trick described in *The Jinx*, a popular magic magazine in its days.

References

[1] Martin Gardner, *Mathematics, Magic and Mystery*, Dover, New York, NY, 1956.
[2] Martin Gardner, Mathematical games: The multiple fascinations of the Fibonacci sequence, *Scientific American*, **220.3** (March 1969), 116–123.

Correction: Ian Fultz's last name was misspelled in the list of solvers of Problem B-1325 in the February issue. The section editor would like to apologize to Ian for his misstep.