ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-833 Proposed by Robert Frontczak, Stuttgart, Germany

The Tribonacci-Lucas numbers $\{K_n\}_{n\geq 0}$ satisfy $K_0 = 3$, $K_1 = 1$, $K_2 = 3$, and $K_n = K_{n-1} + K_{n-2} + K_{n-3}$ for $n \geq 3$. Prove that for any $n \geq 1$

$$\sum_{j=1}^{n} K_{2j}K_{2j+1} = \frac{1}{4}((K_{2n} + K_{2n+1})^2 - 16).$$

H-834 Proposed by Robert Frontczak, Stuttgart, Germany

Let $\{B_n\}_{n\in\mathbb{Z}}$ and $\{C_n\}_{n\in\mathbb{Z}}$ denote the balancing and Lucas-balancing numbers, respectively, given by

$$B_{n+1} = 6B_n - B_{n-1} \quad \text{and} \quad C_{n+1} = 6C_n - C_{n-1} \quad \text{for all} \quad n \geq 1,$$

with $B_0 = 0$, $B_1 = 1$, $C_0 = 1$, and $C_1 = 3$. Prove that for integers $n \geq 1$ and $j \geq 0$

(i) $\sum_{k=1}^{n} C_{k\mp j}B_{k\pm j} = \frac{1}{32}(C_{2n+1} - 3) \pm \frac{n}{2}B_{2j}$;

(ii) $\sum_{k=1}^{n} C_{k-j}C_{k+j}B_{k-j}B_{k+j} = \frac{1}{768}(B_{4n+2} - 6(2n+1)) - \frac{n}{4}B_{2j}^2$.

H-835 Proposed by Andrei K. Svinin and Svetlana V. Svinina, Matrosov Institute for System Dynamics and Control Theory of SB RAS, Russia

Let $B_q^{(k)}$ be the higher order Bernoulli numbers that are defined by an exponential generating function as

$$\frac{t^k}{(e^t - 1)^k} = \sum_{q\geq 0} \frac{B_q^{(k)}}{q!} t^q.$$

Prove that

$$B_n^{(k)} = \sum_{q=1}^{n} \frac{s(q+k,k)}{\binom{q+k}{k}} S(n,q),$$

where $s(n,k)$ and $S(n,k)$ are the Stirling numbers of the first and second type, respectively.

H-836 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Given a real number $p > 0$, define the sequence $\{S_n\}_{n\geq 0}$ by

$$S_0 = p, \quad S_n = S_{n-1}^2 + p \quad \text{for} \quad n \geq 1.$$

For any integer $n \geq 0$, find closed form expressions for the sums

$$\text{(i)} \quad \sum_{k=0}^{n} S_k S_{k+1} \cdots S_n \qquad \text{and} \qquad \text{(ii)} \quad \sum_{k=0}^{n} (S_k S_{k+1} \cdots S_n)^2.$$

SOLUTIONS

A forgotten problem

H-500 Proposed by H.-J. Seiffert, Berlin, Germany
(Vol. 33, No. 4, August 1995)

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ for $n \geq 2$. Show that for all complex numbers x and all nonnegative integers n,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n+2}{n-2k} F_{2k+1}(x) = x^n F_{n+1}(4/x), \tag{1}$$

where $\lfloor\ \rfloor$ denotes the greatest integer function. As special cases, we obtain the following identities

$$\begin{aligned}
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n+2}{n-2k} F_{2k+1} &= \frac{1}{2} F_{3n+3}; \\
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n+2}{n-2k} F_{6k+3} &= 2^{2n+1} F_{n+1}; \\
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n+2}{n-2k} L_{4k+2} &= \frac{1}{2}(5^{n+1} - (-1)^{n+1}).
\end{aligned}$$

Solution by Ulrich Abel and Vitaliy Kushnirevych

The recursive formula $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ with $F_0(x) = 0,\ F_1(x) = 1$ of the Fibonacci polynomials $F_n(x)$ yields the generating function

$$\sum_{n=0}^{\infty} F_n(x) z^n = \frac{z}{1 - xz - z^2},$$

which implies the well-known explicit representation

$$F_{n+1}(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} x^{n-2j}.$$

We have

$$\begin{aligned}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2n+2}{n-2k}F_{2k+1}(x) &= \sum_{k=0}^{\lfloor n/2\rfloor}\binom{2n+2}{n-2k}\sum_{j=0}^{k}\binom{2k-j}{j}x^{2k-2j}\\ &= \sum_{k=0}^{\infty}\binom{2n+2}{n+2+2k}\sum_{j=0}^{k}\binom{k+j}{k-j}x^{2j}\\ &= \sum_{j=0}^{\infty}x^{2j}\sum_{k=j}^{\infty}\binom{2n+2}{n+2+2k}\binom{k+j}{k-j}.\end{aligned}$$

As usual, for a formal power series, we write $[z^n]\sum_{n=0}^{\infty}a_nz^n$ to denote the coefficient a_n of z^n. Noting that

$$\binom{2n+2}{n+2+2k}=\binom{2n+2}{n-2k}=[z^n]\frac{z^{2k}}{(1-z)^{2k+n+3}},$$

we see that the inner sum is equal to

$$\begin{aligned}[z^n]\sum_{k=0}^{\infty}\binom{k+2j}{k}\frac{z^{2k+2j}}{(1-z)^{2k+2j+n+3}} &= [z^n]\frac{z^{2j}}{(1-z)^{2j+n+3}}\left(1-\frac{z^2}{(1-z)^2}\right)^{-(2j+1)}\\ &= \left[z^{n-2j}\right]\frac{1}{(1-z)^{n+2}(1-2z)^{2j+1}}.\end{aligned}$$

Application of the Lagrange inversion

$$[z^n]\left[F(w)\mid w=z\varphi(w)\right]=[z^n]\left[F(z)\varphi^{n-1}(z)\left(\varphi(z)-z\varphi'(z)\right)\right],$$

with $\varphi(z)=1/(1-z)$ and $F(z)=1/(1-2z)^{2j+2}$ yields

$$\begin{aligned}\left[z^{n-2j}\right]\frac{1}{(1-z)^{n+2}(1-2z)^{2j+1}} &= \left[z^{n-2j}\right]\left[F(w)\mid w=z/(1-w)\right]\\ &= \left[z^{n-2j}\right]\frac{1}{\left(\mp\sqrt{1-4z}\right)^{2j+2}}=\left[z^{n-2j}\right]\frac{1}{(1-4z)^{j+1}}\\ &= 4^{n-2j}\binom{n-j}{j}.\end{aligned}$$

This implies the desired identity because the right side is equal to

$$x^nF_{n+1}(4/x)=x^n\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}(4/x)^{n-2j}=\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}4^{n-2j}x^{2j}.$$

This proves (1).

Some divisibility relations with members of Lucas sequences

H-801 Proposed by Refik Keskin, Sakarya University, Turkey and Florian Luca, University of the Witwatersrand, Johannesburg, South Africa (Vol. 55, No. 1, February 2017)

Let $P\geq 3$ be an integer and $(V_n)_{n\geq 0}$ be the sequence given by $V_0=2$, $V_1=P$, and $V_{n+2}=PV_{n+1}-V_n$ for $n\geq 0$. Assume that $3\nmid n$. Show that:

(i) $P+1\mid V_n+1$;

(ii) If $V_n + 1 = (P+1)F(P)$, then $F(-1) = n$ if $n \equiv 1 \pmod 3$ and $F(-1) = -n$ if $n \equiv 2 \pmod 3$.

Solution by Eduardo H. M. Brietzke

Consider the generating function

$$g(x) := \sum_{n=0}^{\infty} V_n x^n = \frac{2 - Px}{1 - Px + x^2}.$$

We have

$$f(x) := \sum_{n=0}^{\infty} (V_n + 1)x^n = \frac{2 - Px}{1 - Px + x^2} + \frac{1}{1-x} = \frac{3 - 2(P+1)x + (P+1)x^2}{1 - (P+1)x + (P+1)x^2 - x^3}.$$

Let $\omega := e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then,

$$\sum_{n=0}^{\infty} V_{3n+1} x^{3n+1} = \frac{f(x) + \omega^2 f(\omega x) + \omega f(\omega^2 x)}{3} = (P+1)xh(x,P),$$

where

$$h(x,P) := \frac{1 - (P^2-2)x^3 + (P-1)x^6}{1 - (P^3 - 3P + 1)x^3 + (P^3 - 3P + 1)x^6 - x^9}$$

(The above calculation can be done by hand or by using a symbolic calculation package). Set

$$\sum_{n=0}^{\infty} F_n(P)x^n := xh(x,P).$$

It is easy to check that

$$\sum_{n=0}^{\infty} F_n(-1)x^n = xh(x,-1) = \frac{x(1 + x^3 - 2x^6)}{(1-x^3)^3} = \sum_{n=0}^{\infty} (3n+1)x^{3n+1}. \tag{2}$$

The last equality follows from the expansion

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n,$$

from which it follows that

$$\begin{aligned} \frac{1 + x - 2x^2}{(1-x)^3} &= \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n + \sum_{n=0}^{\infty} \frac{(n+1)n}{2} x^n + 2\sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n \\ &= \sum_{n=0}^{\infty} (3n+1)x^n. \end{aligned}$$

Equality (2) proves that if $n \equiv 1 \pmod 3$, then $P + 1 \mid V_n + 1$ and $V_n + 1 = n(P+1)$.

The case $n \equiv 2 \pmod 3$ is similar. We have

$$\sum_{n=0}^{\infty} V_{3n+2} x^{3n+2} = \frac{f(x) + \omega f(\omega x) + \omega^2 f(\omega^2 x)}{3} = (P+1)x^2 k(x,P),$$

with

$$k(x,P) := \frac{(P-1) - (P^2-2)x^3 + x^6}{1 - (P^3 - 3P + 1)x^3 + (P^3 - 3P + 1)x^6 - x^9}.$$

Set

$$\sum_{n=0}^{\infty} G_n(P)x^n := x^2 k(x,P).$$

Then,

$$x^2 k(x,-1) = \frac{x^2(-2+x^3+x^6)}{(1-x^3)^3} = -\sum_{n=0}^{\infty}(3n+2)x^{3n+2}.$$

Hence, if $n \equiv 2 \pmod 3$, then $V_n + 1 = -n(P+1)$.

Also solved by Dmitry G. Fleischman and the proposers.

An inequality with Fibonacci numbers

H-802 **Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 55, No. 1, February 2017)**

Let a, b, c, d be positive integers such that $a \geq b$, $c \geq d$, and b and d have the same parity. Then for all integers $n \geq 1$, prove that

$$\left(\sum_{k=1}^{n} F_{F_k+a}F_{L_k+c}\right)\left(\sum_{k=1}^{n} F_{F_k+b}F_{L_k+d}\right) \geq \left(\sum_{k=1}^{n} F_{F_k+a}F_{L_k+d}\right)\left(\sum_{k=1}^{n} F_{F_k+b}F_{L_k+c}\right).$$

Solution by the proposer

For $n = 1$, the inequality clearly holds. Let $n \geq 2$. We use the following identity. For $s+t = u+v$,

$$F_sF_t - F_uF_v = (-1)^r(F_{s-r}F_{t-r} - F_{u-r}F_{v-r}) \tag{3}$$

(see [1]). Using the Binet-Cauchy inequality, we have

$$\begin{aligned} LS - RS &= \sum_{1\leq i<j\leq n} (F_{F_i+a}F_{F_j+b} - F_{F_j+a}F_{F_i+b})(F_{L_i+c}F_{L_j+d} - F_{L_j+c}F_{L_i+d}) \\ &= \sum_{1\leq i<j\leq n} (-1)^{F_i+b}F_{a-b}F_{F_j-F_i} \cdot (-1)^{L_i+d}F_{c-d}F_{L_j-L_i} \qquad \text{by (3)} \\ &= \sum_{1\leq i<j\leq n} (-1)^{F_i+L_i}(-1)^{b+d}F_{a-b}F_{c-d}F_{F_j-F_i}F_{L_j-L_i} \geq 0. \end{aligned}$$

[1] R. C. Johnson, *Fibonacci numbers and matrices*, http://www.dur.ac.uk/bob.johnson/fibonacci/

Also partially solved by Dmitry G. Fleischman.

The area of a polygon with Lucas number coordinates

H-803 **Proposed by Ángel Plaza, Gran Canaria, Spain (Vol. 55, No. 1, February 2017)**

Assume that the consecutive numbers in the Lucas sequence are coordinates of the vertices of a polygon labeled counterclockwise in the Cartesian system:

$$A_1(L_1,L_2);\ A_2(L_3,L_4);\ A_3(L_5,L_6);\ \ldots;\ A_n(L_{2n-1},L_{2n}).$$

What is the area of such a polygon?

Solution by the proposer

Since the polygon's vertices in the Cartesian coordinate system are counterclockwise, we may apply the Shoelace formula [1]. That is: if the vertices of a simple polygon, listed counterclockwise around the perimeter, are (x_0, y_0), (x_1, y_1), …, (x_{n-1}, y_{n-1}), the area of the polygon is

$$A = \frac{1}{2}\left| \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-2} & x_{n-1} \\ y_{n-2} & y_{n-1} \end{vmatrix} + \begin{vmatrix} x_{n-1} & x_0 \\ y_{n-1} & y_0 \end{vmatrix} \right|.$$

Our case reads as

$$\begin{aligned} A &= \frac{1}{2}\left| \begin{vmatrix} L_1 & L_3 \\ L_2 & L_4 \end{vmatrix} + \begin{vmatrix} L_3 & L_5 \\ L_4 & L_6 \end{vmatrix} + \cdots + \begin{vmatrix} L_{2n-3} & L_{2n-1} \\ L_{2n-2} & L_{2n} \end{vmatrix} + \begin{vmatrix} L_{2n-1} & L_1 \\ L_{2n} & L_2 \end{vmatrix} \right| \\ &= \frac{5F_{2n-2} - 5(n-1)}{2}, \end{aligned}$$

where the last identity follows since for $k = 2, \ldots, n$ we have

$$\begin{vmatrix} L_{2k-3} & L_{2k-1} \\ L_{2k-2} & L_{2k} \end{vmatrix} = L_{4k-3} - L_3 - L_{4k-3} - L_1 = -5$$

and for the last determinant

$$\begin{vmatrix} L_{2n-1} & 1 \\ L_{2n} & 3 \end{vmatrix} = 3L_{2n-1} - L_{2n} = 5F_{2n-2}.$$

□

[1] B. Braden, *The surveyor's area formula*, The College Mathematics Journal, **17**(4) (1986), 326–337.

Also solved by Dmitry G. Fleischman, Wei-Kai Lai and John Risher (jointly), Jason Smith and David Terr.

Sums of reciprocals of products of Lucas numbers

H-804 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 55, No. 1, February 2017)

Prove that

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\alpha^{n(n-3)} L_2 L_4 L_6 \cdots L_{2n}} = 1;$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\alpha^{2F_{n-1}} L_{2F_1} L_{2F_2} L_{2F_3} \cdots L_{2F_n}} = \frac{1}{\alpha^2};$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\alpha^{2L_{n-1}} L_{2L_1} L_{2L_2} L_{2L_3} \cdots L_{2L_n}} = \frac{1}{\alpha^6}.$

Solution by the proposer

We need the following lemma.

Lemma For all positive integer sequences $\{a_n\}_{n\geq 1}$, putting $S_n = \sum_{i=1}^{n} a_i$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{S_n + a_n}}{\alpha^{S_n - 2a_n} L_{a_1} L_{a_2} \cdots L_{a_n}} = 1.$$

Proof. Let $b_n = (-\alpha^2)^{a_n} + 1$. We have

$$\begin{aligned}\sum_{n=1}^{m} \frac{b_n - 1}{b_1 b_2 \cdots b_n} &= \frac{b_1 - 1}{b_1} + \sum_{n=2}^{m} \left(\frac{1}{b_1 b_2 \cdots b_{n-1}} - \frac{1}{b_1 b_2 \cdots b_n} \right) \\ &= 1 - \frac{1}{b_1} + \frac{1}{b_1} - \frac{1}{b_1 b_2 \cdots b_m} \to 1 \qquad (\text{as} \quad m \to \infty).\end{aligned}$$

Therefore, we have

$$\begin{aligned}1 &= \sum_{n=1}^{\infty} \frac{(-\alpha^2)^{a_n}}{\prod_{i=1}^{n}((-\alpha^2)^{a_i} + 1)} = \sum_{n=1}^{\infty} \frac{(-\alpha^2)^{a_n}}{\prod_{i=1}^{n}(-\alpha)^{a_i} \prod_{i=1}^{n}(\alpha^{a_i} + \beta^{a_i})} \\ &= \sum_{n=1}^{\infty} \frac{(-\alpha^2)^{a_n}}{(-\alpha)^{S_n} \prod_{i=1}^{n} L_{a_i}} = \sum_{n=1}^{\infty} \frac{(-1)^{S_n + a_n}}{\alpha^{S_n - 2a_n} \prod_{i=1}^{n} L_{a_i}}.\end{aligned}$$

□

(i) If $a_n = 2n$, then we have

$$S_n - 2a_n = \sum_{i=1}^{n} 2i - 4n = n(n+1) - 4n = n(n-3).$$

By the Lemma, we obtain the desired identity.

(ii) Note that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$. If $a_n = 2F_n$, then we have

$$\begin{aligned}S_n - 2a_n &= \sum_{i=1}^{n} 2F_i - 2F_n = 2F_{n+2} - 2 - 4F_n \\ &= 2(F_{n+2} - 2F_n) - 2 = 2(F_{n+1} - F_n) - 2 = 2F_{n-1} - 2.\end{aligned}$$

By the Lemma, we have

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{2F_{n-1} - 2} L_{2F_1} L_{2F_2} \cdots L_{2F_n}} = 1.$$

Therefore, we obtain the desired identity.

(iii) Note that $\sum_{i=1}^{n} L_i = L_{n+2} - 3$. If $a_n = 2L_n$, then we have

$$\begin{aligned}S_n - 2a_n &= \sum_{i=1}^{n} 2L_i - 4L_n = 2L_{n+2} - 6 - 4L_n \\ &= 2(L_{n+2} - 2L_n) - 6 = 2(L_{n+1} - L_n) - 6 = 2L_{n-1} - 6.\end{aligned}$$

By the Lemma, we have

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{2L_{n-1} - 6} L_{2L_1} L_{2L_2} \cdots L_{2L_n}} = 1.$$

Therefore, we obtain the desired identity.

Note: In the same manner, one obtains

$$\sum_{n=1}^{\infty} \frac{(-1)^{S_n + a_n + n}}{\alpha^{S_n - 2a_n} (\sqrt{5})^n F_{a_1} F_{a_2} \cdots F_{a_n}} = 1.$$

Also partially solved by Dmitry G. Fleishman.