ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2019. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-756 Proposed by Stanley Rabinowitz, Chelmsford, MA. (Vol. 32.1, February 1994)

Find a formula expressing the Pell number P_n in terms of Fibonacci and/or Lucas numbers.

Editor's Note: This is an old problem from 1994 that was proposed by the former Problem Section Editor. At that time, no relatively simple and elegant solutions were received, so the editor left the problem open. At its 25th anniversary, we have revived the problem, and invite the readers to solve it.

B-1241 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all positive integers n, prove that

$$\frac{F_{n+2}}{L_{n+2}} + \frac{F_{n+1}}{L_{n+1}} + \frac{F_n}{L_{n+1} + F_{n+2}} > 1.$$

B-1242 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let $r_1, r_2, \ldots, r_n$ be positive even integers. Prove that

$$\sum_{\epsilon_1,\cdots,\epsilon_n \in \{-1,1\}} F_{\epsilon_1 r_1+\cdots+\epsilon_n r_n} = 0, \qquad \text{and} \qquad \sum_{\epsilon_1,\cdots,\epsilon_n \in \{-1,1\}} L_{\epsilon_1 r_1+\cdots+\epsilon_n r_n} = 2\prod_{k=1}^{n} L_{r_k}.$$

B-1243 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

For any positive integer k, the k-Fibonacci numbers are defined recursively by $F_{k,0} = 0$, $F_{k,1} = 1$, and $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$. Prove that

$$k\sum_{m=0}^{n} F_{k,m} = \left\lfloor \frac{(\sqrt{k^2+4}+k)^{n+1} - 2^{n+1}}{2^n\sqrt{k^2+4}\,(\sqrt{k^2+4}-k)} \right\rfloor.$$

B-1244 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Let $n \geq 2$ be an integer. Prove the following identities for the Fibonacci and Lucas numbers:

a) $\displaystyle\sum_{k=1}^{n-1}\sum_{j=k+1}^{n} (F_k - F_j)^2 = nF_nF_{n+1} - (F_{n+2}-1)^2.$

b) $\displaystyle\sum_{k=1}^{n-1}\sum_{j=k+1}^{n} (L_k - L_j)^2 = n(L_nL_{n+1}-2) - (L_{n+2}-3)^2.$

c) $\displaystyle\sum_{k=1}^{n-1}\sum_{j=k+1}^{n} (F_kL_j - F_jL_k)^2 = \begin{cases} F_nF_{n+1}(L_nL_{n+1}-2) - (F_{n+1}L_n - 2)^2, & \text{if } n \text{ is even,} \\ F_nF_{n+1}(L_nL_{n+1}-2) - F_{n+1}^2L_n^2, & \text{if } n \text{ is odd.} \end{cases}$

B-1245 Proposed by Kenny B. Davenport, Dallas, PA.

Show that, for any positive integer n,

$$\sum_{k=1}^{n} kL_k^3 = \frac{5\left[(2n+3)L_{3n-1} - L_{3n}\right] + 49}{4} - (n+2)L_{n-1}^3 - L_n^3.$$

Determinant of a Symmetric Matrix

B-1221 Proposed by José Luis Díaz-Barrero, Technical University of Catalonia (Barcelona Tech), Barcelona Spain. (Vol. 56.1, February 2018)

For any positive integer n, show that

$$\frac{1}{54F_{2n}}\begin{vmatrix} 4 & F_n & L_n \\ F_n & (F_{n+1}+L_n)^2 & F_{2n} \\ L_n & F_{2n} & F_{n+2}^2 \end{vmatrix}$$

is a perfect square, and find its value.

Composite solution by I. V. Fedak, Vasyl Stefanyk Precarpathian National University. Ivano-Frankivsk, Ukraine, and the editor.

Let $x = F_{n-1}$ and $y = F_{n+1}$, so that

$$F_n = y - x, \quad L_n = y + x, \quad F_{n+2} = 2y - x, \quad F_{n+1} + L_n = 2y + x,$$

and

$$F_{2n} = F_n L_n = (y - x)(y + x) = y^2 - x^2.$$

Then,

$$\begin{vmatrix} 4 & F_n & L_n \\ F_n & (F_{n+1} + L_n)^2 & F_{2n} \\ L_n & F_{2n} & F_{n+2}^2 \end{vmatrix} = \begin{vmatrix} 4 & y - x & y + x \\ y - x & (2y + x)^2 & y^2 - x^2 \\ y + x & y^2 - x^2 & (2y - x)^2 \end{vmatrix}$$
$$= 4(2y + x)^2(2y - x)^2 - 2(y^2 - x^2)^2 - (y + x)^2(2y + x)^2 - (y - x)^2(2y - x)^2,$$

which can be simplified to

$$\begin{aligned} &\left[(2(2y - x)^2 - (y + x)^2\right](2y + x)^2 + \left[2(2y + x)^2 - (y - x)^2\right](2y - x)^2 - 2(y^2 - x^2)^2 \\ = \;& (7y^2 - 10xy + x^2)(2y + x)^2 + (7y^2 + 10xy + x^2)(2y - x)^2 - 2(y^2 - x^2)^2 \\ = \;& 2(7y^2 + x^2)(4y^2 + x^2) - 80x^2y^2 - 2(y^2 - x^2)^2 \\ = \;& 54y^2(y^2 - x^2) \\ = \;& 54F_{n+1}^2 F_{2n}. \end{aligned}$$

Thus, the value of the given expression is F_{n+1}^2 for any positive inteber n.

Also solved by Brian D. Beasley, Dmitry Fleischman, G. C. Gruebel, Stacy M. Hartz (student), Wei-Kai Lai, Ehren Metcalfe, Kambiz Moghaddamfar (student), Raphael Schumacher (student), Jaroslav Seibert, Jason L. Smith, Albert Stadler, Nicuşor Zlota, and the proposer.

The Generating Function for Harmonic Numbers

B-1222 Proposed by Kenny B. Davenport, Dallas, PA. (Vol. 56.1, February 2018)

Let H_n denote the nth harmonic number. Prove that

$$\sum_{n=2}^{\infty} \frac{H_{n-1}F_n}{n2^n} = \frac{\ln 16 \cdot \ln \alpha}{\sqrt{5}}, \qquad \text{and} \qquad \sum_{n=2}^{\infty} \frac{H_{n-1}L_n}{n2^n} = (\ln 2)^2 + 4(\ln \alpha)^2.$$

Solution by Amanda M. Andrews and Samantha L. Zimmerman (students), California University of Pennsylvania, California, PA (jointly).

We will deduce the results for the generalized Fibonacci sequence $\{G_n\}_{n\in\mathbb{N}}$ defined by $G_1 = a$, $G_2 = b$, and $G_n = G_{n-1} + G_{n-2}$, for $n \geq 3$. The generating function for H_n is known to be

$$\sum_{n=1}^{\infty} H_n x^n = \frac{1}{1 - x} \ln\left(\frac{1}{1 - x}\right), \qquad x \in (-1, 1).$$

Now, after integrating this power series over $[0,x]$ for $x \in (-1,1)$, we obtain

$$\sum_{n=2}^{\infty} \frac{H_{n-1}x^n}{n} = \frac{1}{2}\ln^2(1-x), \qquad x \in (-1,1).$$

We also use [1, p. 111]

$$G_n = \frac{c\alpha^n - d\beta^n}{\sqrt{5}}, \qquad n \in \mathbb{Z},$$

where $c = a + (a-b)\beta$, and $d = a + (a-b)\alpha$. Since $\frac{\alpha}{2}, \frac{\beta}{2} \in (-1,1)$, we find

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_{n-1}G_n}{n2^n} &= \frac{c}{\sqrt{5}} \sum_{n=2}^{\infty} \frac{H_{n-1}}{n}\left(\frac{\alpha}{2}\right)^n - \frac{d}{\sqrt{5}} \sum_{n=2}^{\infty} \frac{H_{n-1}}{n}\left(\frac{\beta}{2}\right)^n \\ &= \frac{c}{2\sqrt{5}} \ln^2\left(1-\frac{\alpha}{2}\right) - \frac{d}{2\sqrt{5}} \ln^2\left(1-\frac{\beta}{2}\right).\end{aligned}$$

It is easy to verify that $1-\frac{\alpha}{2} = \frac{1}{2\alpha^2}$, and $1-\frac{\beta}{2} = \frac{1}{2\beta^2} = \frac{\alpha^2}{2}$. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_{n-1}G_n}{n2^n} &= \frac{1}{2\sqrt{5}}\left[c\ln^2\left(\frac{1}{2\alpha^2}\right) - d\ln^2\left(\frac{\alpha^2}{2}\right)\right] \\ &= \frac{1}{2\sqrt{5}}\left[c\,(\ln 2 + 2\ln\alpha)^2 - d\,(\ln 2 - 2\ln\alpha)^2\right] \\ &= \frac{1}{2\sqrt{5}}\Big((c-d)\left[(\ln 2)^2 + 4(\ln\alpha)^2\right] + 4(c+d)\ln 2\cdot\ln\alpha\Big).\end{aligned}$$

For Fibonacci numbers, we have $a = b = 1$; hence, $c = d = 1$, and

$$\sum_{n=2}^{\infty} \frac{H_{n-1}F_n}{n2^n} = \frac{1}{2\sqrt{5}}(8\ln 2\cdot\ln\alpha) = \frac{\ln 16\cdot\ln\alpha}{\sqrt{5}}.$$

For Lucas numbers, we have $a = 1$ and $b = 3$; hence, $c = -d = \sqrt{5}$, which leads to $c + d = 0$, and $c - d = 2\sqrt{5}$. Thus,

$$\sum_{n=2}^{\infty} \frac{H_{n-1}L_n}{n2^n} = (\ln 2)^2 + 4(\ln\alpha)^2.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.

Also solved by Khristo N. Boyadzhiev, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Albert Natian, Hikeyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (student), Jaroslav Seibert, Jason L. Smith, Albert Stadler, Santiago Alzate Suárez (student), and the proposer.

An Inequality with a Geometric Twist

B-1223 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 56.1, February 2018)

For all positive integers n and a, prove that

$$\sum_{k=1}^{n} F_k(F_{k+1}^a + F_{k+2}^a - F_{n+2}^a - 1) \le 0.$$

Solution 1 by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, SC.

The claimed inequality is equivalent to

$$\sum_{k=1}^{n} F_k(F_{k+1}^a + F_{k+2}^a) \le (F_{n+2}^a + 1)\sum_{k=1}^{n} F_k = (F_{n+2}^a + 1)(F_{n+2} - 1).$$

We find

$$\sum_{k=1}^{n} F_k(F_{k+1}^a + F_{k+2}^a) = F_1F_2^a + F_nF_{n+2}^a + \sum_{k=1}^{n-1}(F_k + F_{k+1})F_{k+2}^a = 1 + F_nF_{n+2}^a + \sum_{k=1}^{n-1} F_{k+2}^{a+1}.$$

Since $F_1 = F_2 = 1$, we can further rewrite the claimed inequality as

$$F_nF_{n+2}^a + \sum_{i=1}^{n+1} F_i^{a+1} \le F_{n+2}^{a+1} - F_{n+2}^a + F_{n+2},$$

or

$$\sum_{i=1}^{n+1} F_i^{a+1} \le F_{n+2}^a F_{n+1} - F_{n+2}^a + F_{n+2}.$$

We will prove this inequality by induction on n. The equality becomes an equality when $n = 1$. Assume it is true when $n = k$. Then,

$$\sum_{i=1}^{k+2} F_i^{a+1} \le F_{k+2}^a F_{k+1} - F_{k+2}^a + F_{k+2} + F_{k+2}^{a+1}.$$

To complete the inductive step, it suffices to prove that

$$F_{k+2}^a F_{k+1} - F_{k+2}^a + F_{k+2} + F_{k+2}^{a+1} \le F_{k+3}^a F_{k+2} - F_{k+3}^a + F_{k+3},$$

or equivalently,

$$F_{k+1}(F_{k+2}^a - 1) \le (F_{k+2} - 1)(F_{k+3}^a - F_{k+2}^a).$$

After factoring $F_{k+2}^a - 1$ and $F_{k+3}^a - F_{k+2}^a$ and canceling common factors, the inequality above reduces to

$$\sum_{j=0}^{a-1} F_{k+2}^{a-1-j} \le \sum_{j=0}^{a-1} F_{k+3}^{a-1-j} F_{k+2}^j,$$

which is obviously true. Therefore, the claimed inequality is true for any positive integer n.

Solution 2 by the proposer.

The inequality becomes an equality when $n = 1$, so we shall assume $n > 1$. Using $F_2 = 1$, and the identities $F_k = F_{k+2} - F_{k+1}$ and $\sum_{k=1}^{n} F_k = F_{n+2} - 1$, we can write the given inequality as

$$\sum_{k=1}^{n} (F_{k+2} - F_{k+1}) \cdot \frac{F_{k+1}^a + F_{k+2}^a}{2} \le (F_{n+2} - F_2) \cdot \frac{F_{n+2}^a + F_2^a}{2}.$$

Let A_k denote the point $\left(F_k, F_k^a\right)$ on the graph of the function $f(x) = x^a$, and B_k denote the point $(F_k, 0)$. The left side is the sum of the areas of the trapezoids $A_{k+1}A_{k+2}B_{k+2}B_{k+1}$ from $k = 1$ to $k = n$. The right side of the inequality above is the area of the trapezoid $A_2A_{n+2}B_{n+2}B_2$. Because $f(x) = a^x$ is a convex function, it is obvious that the left side is less than or equal to the right side.

Also solved by Dmitry Fleischman, and Ángel Plaza.

An Intriguing Binomial Sum

B-1224 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 56.1, February 2018)

For any positive integer n, prove that

$$\sum_{k=1}^{n}\binom{n}{k}\frac{F_k}{k} = \sum_{k=1}^{n}\frac{F_{2k}}{k}, \qquad \text{and} \qquad \sum_{k=1}^{n}\binom{n}{k}\frac{L_k}{k} = \sum_{k=1}^{n}\frac{L_{2k}-2}{k}.$$

Solution 1 by Kambiz Moghaddamfar (student), Sharif University of Technology, Tehran, Iran.

Given G_0 and G_1, the generalized Fibonacci sequence $G_0, G_1, G_2, \ldots,$ is defined recursively by $G_n = G_{n-1} + G_{n-2}$ for $n \geq 2$. First, we claim that

$$\sum_{k=1}^{n}\frac{G_{2k}}{k} = G_0\left(\sum_{k=1}^{n}\frac{1}{k}\right) + \sum_{i=1}^{n}\frac{G_i}{i}\binom{n}{i}.$$

To prove this, we apply the well-known identity [1] that

$$G_{2k} = \sum_{i=0}^{k}\binom{k}{i}G_i$$

to obtain

$$\begin{aligned}
\sum_{k=1}^{n}\frac{G_{2k}}{k} &= \sum_{k=1}^{n}\frac{1}{k}\left[\sum_{i=0}^{k}\binom{k}{i}G_i\right]\\
&= G_0\left(\sum_{k=1}^{n}\frac{1}{k}\right) + \sum_{i=1}^{n}G_i\left[\sum_{k=i}^{n}\frac{1}{k}\binom{k}{i}\right]\\
&= G_0\left(\sum_{k=1}^{n}\frac{1}{k}\right) + \sum_{i=1}^{n}\frac{G_i}{i}\left[\sum_{j=0}^{n-i}\frac{i}{i+j}\binom{i+j}{i}\right]\\
&= G_0\left(\sum_{k=1}^{n}\frac{1}{k}\right) + \sum_{i=1}^{n}\frac{G_i}{i}\left[\sum_{j=0}^{n-i}\binom{i+j-1}{i-1}\right].
\end{aligned}$$

Applying the Hockey-Stick Theorem, we find

$$\sum_{j=0}^{n-i}\binom{i+j-1}{i-1} = \binom{n}{i},$$

from which the claim follows. The proof is completed by substituting in $G_n = F_n$ and $G_n = L_n$, respectively.

Solution 2 by Khristo N. Boyadzhiev, Ohio Northern University, Ada, OH.

Let a_i, $i = 1, 2, 3, \ldots$, be a sequence, and let

$$b_n = \sum_{k=1}^{n} \binom{n}{k} a_k.$$

The following result was proved in [2, 3]:

$$\sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k} = \sum_{k=1}^{n} \frac{b_k}{k}.$$

Using the Binet's formula and the binomial theorem, it is easy to show that

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}, \qquad \text{and} \qquad \sum_{k=0}^{n} \binom{n}{k} L_k = L_{2n}.$$

Since $F_0 = 0$, we immediately obtain

$$\sum_{k=1}^{n} \binom{n}{k} \frac{F_k}{k} = \sum_{k=1}^{n} \frac{F_{2k}}{k}.$$

The second identity follows in a similar manner, because $L_0 = 2$, and $\sum_{k=1}^{n} \binom{n}{k} L_k = L_{2n} - 2$.

References

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Ratio*, Dover, 2008.
[2] K. N. Boyadzhiev, Binomial transform and the backward difference, *Advan. Appl. Discrete Math.*, **13** (2014), 46–63.
[3] A. N. 't Woord, Solution II to Problem 10490, *Amer. Math. Monthly*, **106** (1999), 588.

Also solved by I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Albert Natian, Ángel Plaza, Raphael Schumacher (student), Jason L. Smith, Albert Stadler, and the proposer.

A Sequence of Matrices with Special Properties

B-1225 Proposed by Jathan Austin, Salisbury University, Salisbury, MD. (Vol. 56.1, February 2018)

Construct a sequence $\{M_n\}_{n=1}^{\infty}$ of 3×3 matrices with positive entries that satisfy the following conditions:

(A) $|M_n|$ is the product of nonzero Fibonacci numbers.
(B) The determinant of any 2×2 submatrix of M_n is a Fibonacci number or the product of nonzero Fibonacci numbers.
(C) $\lim_{n \to \infty} |M_{n+1}|/|M_n| = 1 + 2\alpha$.

Solution by Ehren Metcalfe, Barrie, Ontario, Canada.

Define a sequence of 3×3 matrices $\{M_n\}_{n=1}^{\infty}$ such that

$$M_n = \begin{bmatrix} F_{n+1} & F_{n+2} & F_{n+1} \\ F_{n+2} & F_{n+1} & F_{n+2} \\ F_{n+2} & F_{n+2} & F_{n+1} \end{bmatrix}.$$

Then, each entry of M_n is positive, and

$$\begin{aligned} M_n &= F_{n+1}^3 - F_{n+1}^2 F_{n+2} - F_{n+1} F_{n+2}^2 + F_{n+2}^3 \\ &= (F_{n+1}^2 - F_{n+2}^2)(F_{n+1} - F_{n+2}) \\ &= (F_{n+1} - F_{n+2})^2 (F_{n+1} + F_{n+2}) \\ &= F_n^2 F_{n+3} \end{aligned}$$

is a product of nonzero Fibonacci numbers. For the lower left 2×2 submatrix,

$$\begin{vmatrix} F_{n+2} & F_{n+1} \\ F_{n+2} & F_{n+2} \end{vmatrix} = F_{n+2}^2 - F_{n+2}F_{n+1} = (F_{n+2} - F_{n+1})F_{n+2} = F_n F_{n+2}.$$

For the upper right 2×2 submatrix,

$$\begin{vmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_{n+2} \end{vmatrix} = F_{n+2}^2 - F_{n+1}^2 = (F_{n+2} - F_{n+1})(F_{n+2} + F_{n+1}) = F_n F_{n+3}.$$

Since the upper left and lower right 2×2 can be obtained from the the 2×2 submatrix above by interchanging their rows, their determinants differ by a factor of $-1 = F_{-2}$. All these determinants are products of nonzero Fibonacci numbers. Finally,

$$\lim_{n\to\infty} \frac{|M_{n+1}|}{|M_n|} = \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right)^2 \cdot \frac{F_{n+4}}{F_{n+3}} = \alpha^2 \cdot \alpha = (1+\alpha)\cdot\alpha = 1 + 2\alpha,$$

as desired.

Editor's Note: There are other possible answers. Fedak gave $\begin{bmatrix} F_{n+4} & F_n & F_n \\ F_{n+2} & F_n & F_{n+2} \\ F_n & F_n & F_{n+3} \end{bmatrix}$, and the proposer presented $\begin{bmatrix} 1 & 1 & 1 \\ F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \end{bmatrix}$ as solutions.

Also solved by I. V. Fedak, and the proposer.