ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-717 Proposed by N. Gauthier, The Royal Military College of Canada, Kingston, ON

Let n be a nonnegative integer and let C_n be the nth Catalan number. Prove the following identity:

$$\sum_{k=0}^{n} k^3 C_{n-k} C_k = \frac{n}{2}((n^2+3n+3)C_{n+1} - 3\cdot 4^n).$$

H-717 Proposed by Samuel G. Moreno, Jaén, Spain

Prove that if p is a polynomial such that $p(x) > 0$ for all $x \in \mathbb{R}$, then

$$\sum_{k=0}^{\deg(p)} F_{k+1} y^k p^{(k)}(x) > 0 \qquad \text{for all} \qquad x, y \in \mathbb{R}.$$

H-718 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let $A_{n,m} = F_{n+m}^{2n-2m-3}(F_{n+m}^4 - F_{n-m}^4)$. Prove that

(1) $\displaystyle\prod_{k=2m}^{2n} F_k \le A_{n,m}$ for $n \ge m \ge 1$;

(2) $\displaystyle\prod_{k=m}^{n} F_{2k} < \sqrt{A_{n,m}} \sqrt[4]{\frac{F_{2m-1}^3 F_{2n-1} F_{2n}}{F_{2m-3} F_{2m-2} F_{2n+1}}}$ for $n \ge m \ge 2$.

H-719 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let $T_j(n) = (-1)^{n(j+1)}(F_nF_{n+1})^j$. Given a positive integer m prove that there are rational numbers $\lambda_1, \dots, \lambda_m$ such that

$$\sum_{k=1}^{n}(-1)^{k(m+1)}F_k^{2m} = \sum_{j=1}^{m}\lambda_j T_j(n).$$

Show the identities

(1) $\sum_{k=1}^{n}(-1)^k F_k^4 = -\frac{2}{3}T_1(n) + \frac{1}{3}T_2(n);$

(2) $\sum_{k=1}^{n} F_k^6 = \frac{1}{2}T_1(n) - \frac{1}{4}T_2(n) + \frac{1}{4}T_3(n);$

(3) $\sum_{k=1}^{n}(-1)^k F_k^8 = -\frac{8}{21}T_1(n) + \frac{4}{21}T_2(n) - \frac{2}{7}T_3(n) + \frac{1}{7}T_4(n).$

SOLUTIONS

A Convergent Sequence

H-693 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 47, No. 4, November 2009/2010)

Given a positive integer m prove that the following sequence converges

$$\left\{\sum_{k=1}^{n}\sqrt[m]{F_k} - \sum_{i=1}^{m}\sqrt[m]{F_{n+m+i}}\right\}_{n\geq 1}.$$

Solution by Zbigniew Jakubczyk, Warsaw, Poland.

We use the fact that the sequence $\{a_n\}_{n\geq 1}$ converges if and only if the series $\sum_{n\geq 2}(a_n - a_{n-1})$ is convergent. For us,

$$a_n - a_{n-1} = \sqrt[m]{F_n} + \sqrt[m]{F_{n+m}} - \sqrt[m]{F_{n+2m}}.$$

We obtain from Taylor's formula

$$\sqrt[m]{1+x} = 1 + \frac{x}{m} + o(x) \qquad \text{for} \qquad |x| < 1,$$

and the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \qquad \text{for all} \qquad n \geq 0,$$

where $(\alpha, \beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$, that

$$\sqrt[m]{F_n} = \sqrt[m]{\frac{\alpha^n - \beta^n}{\sqrt{5}}} = \sqrt[m]{\frac{\alpha^n}{\sqrt{5}}\left(1 - \left(\frac{\beta}{\alpha}\right)^n\right)} = \frac{\alpha^{n/m}}{\sqrt[2m]{5}}\sqrt[m]{1 - \left(\frac{\beta}{\alpha}\right)^n}$$
$$= \frac{\alpha^{n/m}}{\sqrt[2m]{5}}\left(1 - \frac{1}{m}\left(\frac{\beta}{\alpha}\right)^n + o\left(\left(\frac{\beta}{\alpha}\right)^n\right)\right).$$

So,

$$\begin{aligned}
a_n - a_{n-1} &= \frac{\alpha^{n/m}}{2^m\sqrt{5}}\left(1 - \frac{1}{m}\left(\frac{\beta}{\alpha}\right)^n + o\left(\left(\frac{\beta}{\alpha}\right)^n\right)\right) \\
&\quad + \frac{\alpha^{(n+m)/m}}{2^m\sqrt{5}}\left(1 - \frac{1}{m}\left(\frac{\beta}{\alpha}\right)^{n+m} + o\left(\left(\frac{\beta}{\alpha}\right)^{n+m}\right)\right) \\
&\quad - \frac{\alpha^{(n+2m)/m}}{2^m\sqrt{5}}\left(1 - \frac{1}{m}\left(\frac{\beta}{\alpha}\right)^{n+2m} + o\left(\left(\frac{\beta}{\alpha}\right)^{n+2m}\right)\right) \\
&= \frac{\alpha^{n/m}}{2^m\sqrt{5}}(1+\alpha-\alpha^2) + \frac{\alpha^{n/m}}{2^m\sqrt{5}}\frac{(\beta/\alpha)^n}{m}\left(-1 - \alpha\left(\frac{\beta}{\alpha}\right)^m + \alpha^2\left(\frac{\beta}{\alpha}\right)^{2m}\right) \\
&\quad + o\left(\frac{\alpha^{n/m}}{2^m\sqrt{5}}\frac{(\beta/\alpha)^n}{m}\left(1 + \alpha\left(\frac{\beta}{\alpha}\right)^m + \alpha^2\left(\frac{\beta}{\alpha}\right)^{2m}\right)\right).
\end{aligned}$$

Note that $1+\alpha-\alpha^2 = 0$, so

$$a_n - a_{n-1} = A_m\left(\alpha^{1/m}\frac{\beta}{\alpha}\right)^n + o\left(\left(\alpha^{1/m}\frac{\beta}{\alpha}\right)^n\right),$$

where

$$A_m = \frac{1}{2^m\sqrt{5}m}\left(-1 - \alpha\left(\frac{\beta}{\alpha}\right)^m + \alpha^2\left(\frac{\beta}{\alpha}\right)^{2m}\right)$$

is a constant that depends only on m. Hence,

$$a_n - a_{n-1} = A_m(1+o(1))\left(\frac{\beta}{\alpha^{1-1/m}}\right)^n.$$

The series

$$\sum_{n\ge 2} A_m\left(\frac{\beta}{\alpha^{1-1/m}}\right)^n$$

is a geometric series with a ratio of $r = \beta/\alpha^{1-1/m}$. As $|r| < 1$, this series is convergent, which completes the solution.

Also solved by Paul S. Bruckman and the proposer.

An Inequality with Products of Fibonacci Numbers

H-694 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 47, No. 4, November 2009/2010)

Prove that the inequality

$$\frac{F_{2n-1}F_{2n+1}}{2} \le \left(\prod_{k=1}^{n}\frac{F_{2k}}{F_{2k-1}}\right)^4 \le \frac{F_{2n-1}F_{2n+2}}{3}$$

holds for all $n \ge 1$.

Solution by the proposer.

First, we show the following inequalities. For a positive integer n,

$$(1)\quad F_{2n+2}^4F_{2n-1} > F_{2n+1}^4F_{2n+3} \quad\text{and}\quad (2)\quad F_{2n+1}^5F_{2n+4} > F_{2n+2}^5F_{2n-1}.$$

For (1), we have

$$\begin{aligned}
&F_{2n+2}^4F_{2n-1} - F_{2n+1}^4F_{2n+3}\\
&\quad= (F_{2n}F_{2n+1}F_{2n+3}F_{2n+4} + 1)F_{2n-1} - (F_{2n-1}F_{2n}F_{2n+2}F_{2n+3} + 1)F_{2n+3}\\
&\quad= F_{2n-1}F_{2n}F_{2n+1}F_{2n+3}F_{2n+4} + F_{2n-1} - F_{2n-1}F_{2n}F_{2n+2}F_{2n+3}^2 - F_{2n+3}\\
&\quad= F_{2n-1}F_{2n}F_{2n+3}(F_{2n+1}F_{2n+4} - F_{2n+2}F_{2n+3}) + F_{2n-1} - F_{2n+3}\\
&\quad= F_{2n-1}F_{2n}F_{2n+3}\cdot(-1)^{2n+3}F_{(2n+1)-(2n+3)} + F_{2n-1} - F_{2n+3}\\
&\quad= F_{2n-1}F_{2n}F_{2n+3} + F_{2n-1} - F_{2n+3}\\
&\quad= F_{2n+3}(F_{2n-1}F_{2n} - 1) + F_{2n-1} > 0.
\end{aligned}$$

In the above calculation, we used the Gelin-Cesàro identity

$$F_n^4 = F_{n-2}F_{n-1}F_{n+1}F_{n+2} + 1 \quad \text{for all} \quad n \geq 2.$$

For (2), we have, similarly as in the proof of (1),

$$\begin{aligned}
&F_{2n+1}^5F_{2n+4} - F_{2n+2}^5F_{2n-1}\\
&\quad= F_{2n+1}^4\cdot F_{2n+1}F_{2n+4} - F_{2n+2}^4\cdot F_{2n+2}F_{2n-1}\\
&\quad= F_{2n+1}F_{2n+4}(F_{2n-1}F_{2n}F_{2n+2}F_{2n+3} + 1) - F_{2n+2}F_{2n-1}(F_{2n}F_{2n+1}F_{2n+3}F_{2n+4} + 1)\\
&\quad= F_{2n+4}F_{2n+1} - F_{2n+2}F_{2n-1} > 0.
\end{aligned}$$

The proof of the following inequalities are by mathematical induction on n.

$$\text{(i)} \quad \prod_{k=1}^{n}\frac{F_{2k}^4}{F_{2k-1}^4} \geq \frac{F_{2n-1}F_{2n+1}}{2} \quad \text{and} \quad \text{(ii)} \quad \prod_{k=1}^{n}\frac{F_{2k}^4}{F_{2k-1}^4} \leq \frac{F_{2n-1}F_{2n+2}}{3}.$$

(i) For $n = 1$, we have $LHS = RHS = 1$. For $n + 1$, assuming that the inequality is true for n, we have

$$\begin{aligned}
\prod_{k=1}^{n+1}\frac{F_{2k}^4}{F_{2k-1}^4} &= \frac{F_{2n+2}^4}{F_{2n+1}^4}\prod_{k=1}^{n}\frac{F_{2k}^4}{F_{2k-1}^4} \geq \frac{F_{2n+2}^4}{F_{2n+1}^4}\cdot\frac{F_{2n-1}F_{2n+1}}{2}\\
&= \frac{F_{2n+2}^4F_{2n-1}}{2F_{2n+1}^3} > \frac{F_{2n+1}^4F_{2n+3}}{2F_{2n+1}^3} = \frac{F_{2n+1}F_{2n+3}}{2},
\end{aligned}$$

where for the last inequality above we used (1). This takes care of (i).

(ii) For $n = 1$, we have $LHS = RHS = 1$. For $n + 1$, assuming that the inequality is true for n, we have

$$\begin{aligned}
\prod_{k=1}^{n+1}\frac{F_{2k}^4}{F_{2k-1}^4} &= \frac{F_{2n+2}^4}{F_{2n+1}^4}\prod_{k=1}^{n}\frac{F_{2k}^4}{F_{2k-1}^4} \leq \frac{F_{2n+2}^4}{F_{2n+1}^4}\cdot\frac{F_{2n-1}F_{2n+2}}{3}\\
&= \frac{F_{2n+2}^5F_{2n-1}}{3F_{2n+1}^4} < \frac{F_{2n+1}^5F_{2n+4}}{3F_{2n+1}^4} = \frac{F_{2n+1}F_{2n+4}}{3},
\end{aligned}$$

where for the last inequality above we used (2). This takes care of (ii), and completes the solution.

Also solved by Paul S. Bruckman.

Ordered Trees and Fibonacci Numbers

H-695 Proposed by Emeric Deutsch, Polytechnic Institute of NYU, Brooklyn, NY
(Vol. 47, No. 4, November 2009/2010)

An *ordered* tree is a rooted tree in which the children of each node form a sequence rather than a tree. The *height* of an ordered tree is the number of edges on a path of maximum length starting at the root. An ordered tree is said to be *symmetric* if it coincides with its reflection in a vertical line passing through the root. Find the number of symmetric ordered trees with n edges and having height at most 3.

Solution by the proposer.

The answer is F_{n+1}. The generating function of all trees of height at most 1 is $1/(1-z)$ (see Figure 1). These trees are symmetric. The generating function of planted trees of height at most 2 is $z/(1-z)$ (see Figure 2). These trees are symmetric. Every tree of height at most 2 is obtained from a finite sequence of the previous trees by joining them at their roots (see Figure 3). Consequently, their generating function is

$$f_2(z) = \frac{1}{1-\frac{z}{1-z}} = \frac{1-z}{1-2z} = 1 + z + 2z^2 + 4z^3 + 8z^4 + 16z^5 + \cdots .$$

Every symmetric tree of height at most 2 is obtained by joining three trees a, b, and c at their roots, where a is any tree of height at most 2, b is either the empty tree or a planted symmetric tree of height at most 2, and c is the reflection of the tree a in a vertical line (Figure 4). Consequently, the generating function of the symmetric trees of height at most 2 is

$$s_2(z) = \left(1 + \frac{z}{1-z}\right) f_2(z^2) = \frac{1+z}{1-2z^2} = 1 + z + 2z^2 + 2z^3 + 4z^4 + 4z^5 + \cdots .$$

(see Figure 5). The generating function of planted trees of height at most 3 is

$$zf_2(z) = \frac{z(1-z)}{1-2z}.$$

Every tree of height at most 3 is obtained from a finite sequence of the previous trees by joining them at their roots (see Figure 6). Consequently, their generating function is

$$f_3(z) = \frac{1}{1-\frac{z(1-z)}{1-2z}} = \frac{1-2z}{1-3z+z^2} = 1 + z + 2z^2 + 5z^3 + 13z^4 + 34z^5 + \cdots .$$

Every symmetric tree of height at most 3 is obtained by joining three trees a', b', and c' at their roots, where a' is any tree of height at most 3, b' is either the empty tree or a planted symmetric tree of height at most 3, and c' is the reflection of the tree a' in a vertical line (Figure 7). Consequently, the generating function of the symmetric trees of height at most 3 is

$$s_3(z) = \left(1 + \frac{z(1+z)}{1-2z^2}\right) f_3(z^2) = \frac{1}{1-z-z^2} = 1 = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + \cdots$$

(see Figure 1), which is what we wanted to prove.

Solver's remark. Obviously, one can find recursively the generating function, according to the number of edges, for the number of symmetric ordered trees having height at most k.

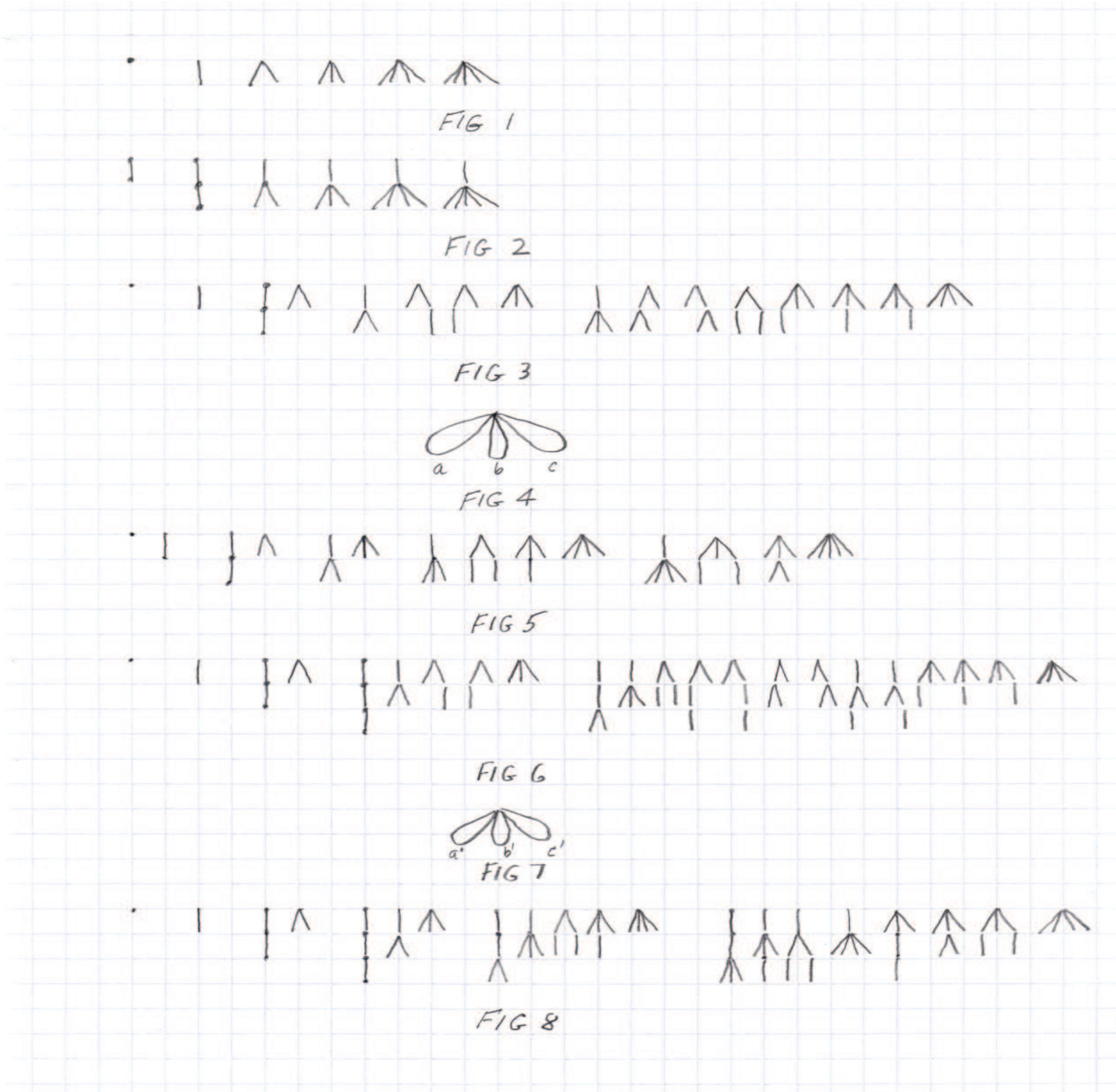

Figure 1. Generating function of the symmetric trees of height at most 3.

Late Acknowledgement: Kenneth B. Davenport has solved H-684 and H-692.