ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-797 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For positive integers a, b, c and $d = a+b+c-1$, prove that

$$\sum_{k=0}^{a} F_{2k} \binom{2a}{a+k}_F \binom{2b}{b+k}_F \binom{2c}{c+k}_F \binom{2d}{d+k}_F^{-1} = \frac{F_a F_b F_c F_{d+1}}{F_{a+b} F_{b+c} F_{c+a}} \binom{2a}{a}_F \binom{2b}{b}_F \binom{2c}{c}_F \binom{2d}{d}_F^{-1}.$$

H-798 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

If $t \in (0, \pi/2)$ and $m \geq 0$ prove that

$$\frac{\sin^{m+2} t}{(F_n \sin t + F_{n+1} \cos t)^m} + \frac{\cos^{m+2} t}{(F_n \cos t + F_{n+1} \sin t)^m} \geq \frac{1}{F_{n+2}^m}$$

and

$$\frac{1}{(L_n + L_{n+1} \tan t)^m} + \frac{\tan^{m+2} t}{(L_n \tan t + L_{n+1})^m} \geq \frac{1}{L_{n+2}^m \cos^2 t}$$

hold for all $n \geq 1$.

H-799 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that

$$\frac{F_n}{F_{n+1}(F_{n+1}^2 + 4F_nF_{n+1} + 3F_n^2)} + \frac{F_{n+1}}{F_n(3F_{n+1}^2 + 4F_nF_{n+1} + F_n^2)} \geq \frac{4F_nF_{n+1}}{F_{n+2}^4}$$

and that the same inequality with all F's replaced by L's holds for all $n \geq 1$.

H-800 **Proposed by Mehtaab Sawhney, Commack, NY.**

Let

$$S_k = \sum_{\substack{n_1+2n_2+\cdots+kn_k=k\\ n_1,n_2,\ldots,n_k\in\mathbb{Z}_{\geq 0}}} (-1)^{n_1+n_2+\cdots+n_k}\binom{n_1+n_2+\cdots+n_k}{n_1,n_2,\ldots,n_k}\prod_{j=1}^{k}(j+1)^{n_j}.$$

Compute $S_1,\ S_2$ and show that $S_k = 0$ for all $k \geq 3$.

SOLUTIONS

Hölder's Inequality in Disguise

H-763 **Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.**
(Vol. 52, No. 4, November 2014)

Prove that:

(i) $\displaystyle\sum_{k=1}^{n}\frac{F_k^4}{k^2} \geq \frac{6F_n^2F_{n+1}^2}{n(n+2)(2n+1)}$;

(ii) $\displaystyle\sum_{k=1}^{n}\frac{F_k^6}{k^2} \geq \frac{4F_n^3F_{n+1}^3}{n^2(n+1)^2}$;

(iii) $\displaystyle\sum_{k=1}^{n}\frac{F_k^6}{k^4} \geq \frac{36F_n^3F_{n+1}^3}{n^2(n+1)^2(2n+1)^2}$;

(iv) $\displaystyle\sum_{k=1}^{n}\frac{F_k^8}{k^3} \geq \frac{4F_n^4F_{n+1}^4}{n^2(n+1)^2}$;

(v) $\displaystyle\sum_{k=1}^{n}\frac{F_k^4}{k^3} \geq \frac{4F_n^2F_{n+1}^2}{n^2(n+1)^2}$;

(vi) $\displaystyle\sum_{k=1}^{n}\frac{F_k^6}{k^6} \geq \frac{16F_n^3F_{n+1}^3}{n^4(n+1)^4}$.

Solution by Hideyuki Ohtsuka.

The inequality (iv) is not correct. We prove (i), (ii), (iii), (v) and (vi).
We use Hölder's inequality

$$\left(\sum_{k=1}^{n}a_k^p\right)^{1/p}\left(\sum_{k=1}^{n}b_k^q\right)^{1/q} \geq \sum_{k=1}^{n}a_kb_k, \tag{1}$$

where $a_k > 0, b_k > 0$ for all $k = 1,\ldots,n$, and $p > 0,\ q > 0$ with $1/p + 1/q = 1$. Let $x_k > 0$ and $y_k > 0$ for $k = 1,\ldots,n$. Letting $a_k = x_k/y_k^{1/q}$ and $b_k = y_k^{1/q}$ in (1), we get

$$\left(\sum_{k=1}^{n}\frac{x_k^p}{y_k^{p/q}}\right)^{1/p}\left(\sum_{k=1}^{n}y_k\right)^{1/q} \geq \sum_{k=1}^{n}x_k.$$

Raising the above inequality to power p we get

$$\left(\sum_{k=1}^{n}\frac{x_k^p}{y_k^{p/q}}\right)\left(\sum_{k=1}^{n}y_k\right)^{p/q} \geq \left(\sum_{k=1}^{n}x_k\right)^p.$$

Since $p/q = p-1$, we obtain

$$\sum_{k=1}^{n} \frac{x_k^p}{y_k^{p-1}} \geq \frac{\left(\sum_{k=1}^{n} x_k\right)^p}{\left(\sum_{k=1}^{n} y_k\right)^{p-1}}. \tag{2}$$

Note that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$.
If $x_k = F_k^2,\ y_k = k^2$ and $p = 2$, by (2) we get (i) since

$$\sum_{k=1}^{n} \frac{F_k^4}{k^2} \geq \frac{6F_n^2 F_{n+1}^2}{n(n+1)(2n+1)} > \frac{6F_n^2 F_{n+1}^2}{n(n+2)(2n+1)}.$$

If $x_k = F_k^2,\ y_k = k$ and $p = 3$, by (2), we obtain (ii).
If $x_k = F_k^2,\ y_k = k^2$ and $p = 3$, by (2), we obtain (iii).
If $x_k = F_k^2,\ y_k = k^3$ and $p = 2$, by (2), we obtain (v).
If $x_k = F_k^2,\ y_k = k^4$, by (2), we obtain (iv).

Editor's comment: In a correspondence of January 12, 2015, Kenneth B. Davenport points out that parts (v) and (vi) are immediate consequences of the published solution to **B1130** in volume **52**, page 183.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, G. C. Greubel, Zbigniew Jakubczyk, Nicuşor Zlota, and the proposers.

Summation Formulas for Fibonomials and Their Squares with Fibonacci Coefficients

H-764 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 52, No. 4, November 2014)

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For $n \geq 1$, prove that

(i) $\displaystyle\sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F = \frac{F_n F_{n+1}}{F_{2n-1}} \binom{2n}{n}_F$;

(ii) $\displaystyle\sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F^2 = \frac{F_n}{L_n} \binom{2n}{n}_F^2$.

Solution by the proposer.

Let s be an even integer. First, we prove the following identities.

(1) $F_{s-1}F_{s-2n-2} + F_n F_{n+1} = F_{s-n-1}F_{s-n-2}$;
(2) $F_s F_{s-2n-2} + F_{n+1}^2 = F_{s-n-1}^2$.

We use the identity

$$F_{r+p}F_{r+q} - F_r F_{r+p+q} = (-1)^r F_p F_q \quad \text{(see [1](20}a\text{))}. \tag{3}$$

(1) Letting $r = s-1$, $p = -n$, and $q = -n-1$ in (3), we have

$$F_{s-n-1}F_{s-n-2} - F_{s-1}F_{s-2n-2} = (-1)^{s-1} F_{-n} F_{-n-1}.$$

Therefore, we get the desired identity.
(2) Letting $r = s$ and $p = q = -n-1$ in (3), we have

$$F_{s-n-1}^2 - F_s F_{s-2n-2} = (-1)^s F_{-n-1}^2.$$

Therefore, we get the desired identity.

(i) For $s > n \geq 0$, we show that

$$\sum_{k=0}^{n} F_{s-2k} \binom{s}{k}_F = \frac{F_{s-n}(F_n + F_{s-n-1})}{F_{s-1}} \binom{s}{n}_F. \tag{4}$$

The proof is by mathematical induction on n. For $n = 0$, we have that both the left and right-hands are F_s. We assume that (4) holds for n. For $n+1$, we have

$$\begin{aligned}
\sum_{k=0}^{n+1} F_{s-2k} \binom{s}{k}_F &= F_{s-2(n+1)} \binom{s}{n+1}_F + \sum_{k=0}^{n} F_{s-2k} \binom{s}{k}_F \\
&= F_{s-2n-2} \binom{s}{n+1}_F + \frac{F_{s-n}(F_n + F_{s-n-1})}{F_{s-1}} \binom{s}{n}_F \\
&= F_{s-2n-2} \binom{s}{n+1}_F + \frac{F_{s-n}(F_n + F_{s-n-1})}{F_{s-1}} \left(\frac{F_{n+1}}{F_{s-n}}\right) \binom{s}{n+1}_F \\
&= \frac{F_{s-1}F_{s-2n-2} + F_n F_{n+1} + F_{s-n-1}F_{n+1}}{F_{s-1}} \binom{s}{n+1}_F \\
&= \frac{F_{s-n-1}F_{s-n-2} + F_{s-n-1}F_{n+1}}{F_{s-1}} \binom{s}{n+1}_F \qquad \text{(by (1))} \\
&= \frac{F_{s-(n+1)}(F_{n+1} + F_{s-(n+1)-1})}{F_{s-1}} \binom{s}{n+1}_F.
\end{aligned}$$

Thus, (4) holds for $n+1$. Therefore, (4) is proved.

Letting $s = 2n$ in (4) for $n \geq 1$, we have

$$\sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F = \frac{F_n(F_n + F_{n-1})}{F_{2n-1}} \binom{2n}{n}_F = \frac{F_n F_{n+1}}{F_{2n-1}} \binom{2n}{n}_F.$$

(ii) For $s > n \geq 0$, we show that

$$\sum_{k=0}^{n} F_{s-2k} \binom{s}{k}_F^2 = \frac{F_{s-n}^2}{F_s} \binom{s}{n}_F^2. \tag{5}$$

The proof is by mathematical induction on n. For $n = 0$, we have that both the left and right-hands are F_s. We assume that (5) holds for n. For $n+1$, we have

$$\begin{aligned}
\sum_{k=0}^{n+1} F_{s-2k} \binom{s}{k}_F^2 &= F_{s-2(n+1)} \binom{s}{n+1}_F^2 + \sum_{k=0}^{n} F_{s-2k} \binom{s}{k}_F^2 \\
&= F_{s-2n-2} \binom{s}{n+1}^2 + \frac{F_{s-n}^2}{F_s} \binom{s}{n}_F^2 \\
&= F_{s-2n-2} \binom{s}{n+1}_F^2 + \left(\frac{F_{s-n}^2}{F_s}\right) \left(\frac{F_{n+1}^2}{F_{s-n}^2}\right) \binom{s}{n+1}_F^2 \\
&= \frac{F_s F_{s-2n-2} + F_{n+1}^2}{F_s} \binom{s}{n+1}_F^2 \\
&= \frac{F_{s-(n+1)}^2}{F_s} \binom{s}{n+1}_F \qquad \text{(by (2))}.
\end{aligned}$$

Thus, (5) holds for $n+1$. Therefore, (5) is proved.

Letting $s = 2n$ in (5) for $n \geq 1$, we have

$$\sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F^2 = \frac{F_n^2}{F_{2n}} \binom{2n}{n}_F^2 = \frac{F_n}{L_n} \binom{2n}{n}_F.$$

References

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

Cauchy-Schwarz to the Rescue

H-765 **Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.**
(Vol. 53, No. 1, February 2015)

Prove that for positive integer n and $m > 0$ we have:

(i) $\dfrac{L_n^4 + L_{n+1}^4}{L_n L_{n+1}} + \dfrac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1} L_{n+3}} + \dfrac{L_{n+3}^4 + L_n^4}{L_{n+3} L_n} \geq \dfrac{2}{3} L_{n+4}^2$;

(ii) $\left(\sum_{k=1}^{n} F_k^{2m+4}\right)\left(\sum_{k=1}^{n} \dfrac{1}{F_k^{2m}}\right) \geq F_n^2 F_{n+1}^2$;

(iii) $\left(\sum_{k=1}^{n} L_k^{2m+4}\right)\left(\sum_{k=1}^{n} \dfrac{1}{L_k^{2m}}\right) \geq (L_n L_{n+1} - 1)^2$;

(iv) $\left(\sum_{k=1}^{n} F_k^{m+2}\right)\left(\sum_{k=1}^{n} \dfrac{1}{F_k^{m}}\right) \geq (F_{n+2} - 1)^2$;

(v) $1 + \sum_{k=1}^{n} \dfrac{F_k^{m+1}}{F_{n-k+1}^m} \geq F_{n+2}$ and $3 + \sum_{k=1}^{n} \dfrac{L_k^{m+1}}{L_{n-k+1}^m} \geq L_{n+2}$.

Solution by Ángel Plaza.

(i) Note that for positive real numbers x, y we have

$$\frac{x^4 + y^4}{xy} \geq x^2 + y^2$$

by Muirhead's inequality. Therefore the left-hand side of the proposed inequality is

$$LHS \geq 2\left(L_n^2 + L_{n+1}^2 + L_{n+3}^2\right).$$

Since $L_{n+4} = L_{n+3} + L_{n+1} + L_n$, the conclusion follows because for $x, y, z > 0$ we have

$$2(x^2 + y^2 + z^2) \geq \frac{2}{3}(x + y + z)^2, \qquad \text{that is} \qquad 3(x^2 + y^2 + z^2) \geq (x + y + z)^2,$$

by the Cauchy-Schwarz inequality.

(ii) By the Cauchy-Schwarz inequality, the left-hand side is

$$LHS \geq \left(\sum_{k=1}^{n} F_k^2\right)^2 = (F_n F_{n+1})^2.$$

(iii) Follows as in (ii) using that

$$\left(\sum_{k=1}^{n} L_k^2\right)^2 = (L_n L_{n+1} - 1)^2.$$

(iv) Follows again by the Cauchy-Schwarz inequality and using that

$$\left(\sum_{k=1}^{n} F_k\right)^2 = (F_{n+2} - 1)^2.$$

(v) This follows by the Chebyshev's sum inequality:

$$\sum_{k=1}^{n} \frac{F_k^{m+1}}{F_{n-k+1}^m} \geq \sum_{k=1}^{n} \frac{F_k^{m+1}}{F_k^m} = \sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

Analogously,

$$\sum_{k=1}^{n} \frac{L_k^{m+1}}{L_{n-k+1}^m} \geq \sum_{k=1}^{n} \frac{L_k^{m+1}}{L_k^m} = \sum_{k=1}^{n} L_k = L_{n+2} - 3.$$

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Hideyuki Ohtsuka, Nicuşor Zlota, and the proposers.

A Quadruple Iterated Sum of Fourth Powers of Fibonacci Numbers

H-766 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 53, No. 1, February 2015)

Let $n = m + 2$. For $m \geq 1$, prove that

$$\sum_{h=1}^{m}\sum_{i=1}^{h}\sum_{j=1}^{i}\sum_{k=1}^{j} F_k^4 = \frac{4F_n^4 + n^4 - 5n^2}{100}.$$

Solution by Zhou Fangmin.

We use the following facts:

(1) $\displaystyle\sum_{n=1}^{\infty} F_n^4 x^n = \frac{x - 4x^2 - 4x^3 + x^4}{1 - 5x - 15x^2 + 15x^3 + 5x^4 - x^5} \triangleq A(x)$,

(2) $\displaystyle\frac{1}{1-x}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} a_m\right) x^n$,

(3) $\displaystyle\sum_{n=0}^{\infty} n^k x^n = \left(x\frac{d}{dx}\right)^k \frac{1}{1-x}$, $k \in \mathbb{N}$.

The desired equation is equivalent to (note that $F_0 = 0$, and

$$4F_1^4 + 1^4 - 5 \cdot 1^2 = 4F_2^4 + 2^4 - 5 \cdot 2^2 = 0),$$

so

$$x^2 \sum_{m\geq 0}\left(\sum_{h=1}^{m}\sum_{i=1}^{h}\sum_{j=1}^{i}\sum_{k=1}^{j} F_k^4\right) x^m = x^2 \sum_{m\geq 0} \frac{4F_n^4 + n^4 - 5n^2}{100} x^m = \sum_{n\geq 0} \frac{4F_n^4 + n^4 - 5n^2}{100} x^n,$$

and

$$x^2 \cdot A(x) \cdot \frac{1}{(1-x)^4} = \frac{1}{100}\left(4A(x) + \left(x\frac{d}{dx}\right)^4 \frac{1}{1-x} - 5\left(x\frac{d}{dx}\right)^2 \frac{1}{1-x}\right),$$

and the above equation can be easily checked manually or by a computer algebra program.

Editor's comment: Helmut Prodinger sent in a one line Maple code which can be used to give a computer assisted proof of the desired identity.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Harris Kwong, Nathan Mcanally, Helmut Prodinger, and the proposer.