A FIBONACCI ARRAY*

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We take $u_0 = 0$, $u_1 = 1$,

$$u_{n+1} = u_n + u_{n-1} \qquad (n \geq 1) ,$$

and define

$$(1) \qquad u_{0,n} = u_n \qquad (n = 0, 1, 2, \cdots)$$

as the 0-th row of the array F. We next put

$$(2) \qquad u_{1,n} = u_{n+2} \qquad (n = 0, 1, 2, \cdots) ,$$

the first row of F. For $r \geq 2$ we define $u_{r,n}$ by means of

$$(3) \qquad u_{r,n} = u_{r-1,n} + u_{r-2,n} \qquad (n = 0, 1, 2, \quad) .$$

Thus $u_{r,n}$ is defined for all r, $n \geq 0$. It follows from the definition that

$$(4) \qquad u_{r,n} = u_{r,n-1} + u_{r,n-2} \qquad (n \geq 2) .$$

Indeed, assuming the truth of (4), we get

$$\begin{aligned} u_{r+1,n} &= u_{r,n} + u_{r-1,n} \\ &= u_{r,n-1} + u_{r,n-2} + u_{r-1,n-1} + u_{r-1,n-2} \\ &= u_{r+1,n-1} + u_{r+1,n-2} . \end{aligned}$$

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*Supported in part by National Science Foundation Grant G16485.

The following table is easily computed

r \ n	0	1	2	3	4	5	6	7	8
0	0	1	1	2	3	5	8	13	21
1	1	2	3	5	8	13	21	34	55
2	1	3	4	7	11	18	29	47	76
3	2	5	7	12	19	31	50	81	131
4	3	8	11	19	30	49	79	128	207
5	5	13	18	31	49	80	129	209	338
6	8	21	29	50	79	129	208	337	545
7	13	34	47	81	128	209	337	546	883
8	21	55	76	131	207	338	545	883	1428

The symmetry property

(5) $$u_{r,n} = u_{n,r}$$

is easily proved by making use of (3) and (4).

We now put

(6) $$f_r(x) = \sum_{n=0}^{\infty} u_{r,n} x^n \quad (r = 0, 1, 2, \ldots) .$$

In particular, it follows from (1) and (2) that

(7) $$f_0(x) = \frac{x}{1 - x - x^2} , \quad f_1(x) = \frac{1 + x}{1 - x - x^2} ,$$

and by (3) we have also

(8) $$f_r(x) = f_{r-1}(x) + f_{r-2}(x) \qquad (r \geq 2) .$$

Using (7) and (8), we prove readily that

(9) $$f_r(x) = \frac{u_r + u_{r+1}x}{1 - x - x^2} \qquad (r \geq 0) .$$

Thus (6) yields

(10) $$u_{r,n} = u_r u_{n+1} + u_{r+1} u_n ,$$

which again implies the truth of (5).

If we put

$$f(x,y) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} u_{r,n} x^r y^n ,$$

then by (9)

$$f(x,y) = \sum_{r=0}^{\infty} \frac{u_r + u_{r+1}y}{1 - y - y^2} x^r = \frac{1}{1 - y - y^2}\left(\frac{x}{1 - x - x^2} + \frac{y}{1 - x - x^2}\right)$$

so that

(11) $$f(x,y) = \frac{x + y}{(1 - x - x^2)(1 - y - y^2)} .$$

We remark that (10) is equivalent to

(12) $$u_{r,n} = u_r u_n + u_{r+n} ,$$

as is easily proved.

It appears from the table that

(13) $$u_{r+1,r-1} - u_{r,r} = (-1)^r \qquad (r \geq 1) .$$

Indeed (13) holds for $r = 1$. Then

$$u_{r+2,r} - u_{r+1,r+1} = \left(u_{r+1,r} + u_{rr}\right) - \left(u_{r+1,r} - u_{r+1,r-1}\right)$$
$$= u_{r,r} - u_{r+1,r-1} = (-1)^{r+1} .$$

Also the relation

$$u_{r+2,r-2} - u_{r,r} = (-1)^{r+1} \qquad (r \geq 2) \tag{14}$$

is suggested; the proof of (14) is similar to the proof of (13).

In the next place we have

$$u_{r+3,r-3} - u_{r,r} = (-1)^r\, 4 \qquad (r \geq 3) . \tag{15}$$

The general formula of which (13), (14), and (15) are special cases is

$$u_{r+s,r-s} - u_{r,r} = (-1)^{r-s+1}\, u_s^2 \qquad (r \geq s) . \tag{16}$$

Indeed it follows from (12) that

$$u_{r+s,r-s} - u_{r,r} = u_{r+s}\, u_{r-s} - u_r^2$$

and (16) is an easy consequence.

For a later purpose we shall require the formula

$$\sum_{r=0}^{n-1} u_{r,r} = \begin{cases} 2u_n^2 & (n \text{ even}) \\ 2u_{n+1}\, u_{n-1} & (n \text{ odd}) \end{cases} . \tag{17}$$

This is equivalent to

$$u_{n-1,n-1} = \begin{cases} 2(u_n^2 - u_n u_{n-2}) = 2u_n u_{n-1} & \text{(n even)} \\ 2(u_{n+1} u_{n-1} - u_{n-1}^2) = 2u_n u_{n-1} & \text{(n odd)} \end{cases},$$

which is in agreement with (10).

In connection with (17) we note that

$$\sum_{r=0}^{\infty} u_{r,r} x^r = \frac{2x}{(1+x)(1-3x+x^2)} . \tag{18}$$

Formulas of this kind are perhaps most easily proved by using the familiar representation

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} ,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} , \qquad \beta = \frac{1-\sqrt{5}}{2} .$$

To illustrate we shall evaluate

$$\sum_{r=0}^{\infty} u_{n+r,r} x^r .$$

Since by (12)

$$u_{n+r,r} = u_{n+r}u_r + u_{n+2r} = \frac{1}{5}\left[2\left(\alpha^{n+2r+1} + \beta^{n+2r+1}\right) - (-1)^r\left(\alpha^n + \beta^n\right)\right],$$

we get

$$\sum_{r=0}^{\infty} u_{n+r,r}x^r = \frac{1}{5}\left(\frac{2\alpha^{n+1}}{1-\alpha^2 x} + \frac{2\beta^{n+1}}{1-\beta^2 x} - \frac{\alpha^n + \beta^n}{1+x}\right)$$

$$= \frac{1}{5}\left(\frac{2(v_{n+1} - v_{n-1}x)}{1 - 3x + x^2} - \frac{v_n}{1 + x}\right) ,$$

where

(19) $$v_n = \alpha^n + \beta^n \quad .$$

It follows that

(20) $$\sum_{r=0}^{\infty} u_{n+r,r}x^r = \frac{1}{5} \frac{(v_{n+1} + v_{n-1})(1 - x^2) + 5v_n x}{(1 + x)(1 - 3x + x^2)} \quad .$$

When $n = 0$, (20) reduces to (18). When $n = 1, 2$ we get

(21) $$\sum_{r=0}^{\infty} u_{r,r+1} x^r = \frac{1 + x - x^2}{(1 + x)(1 - 3x + x^2)} \quad ,$$

(22) $$\sum_{r=0}^{\infty} u_{r,r+2} x^r = \frac{1 + 3x - x^2}{(1 + x)(1 - 3x + x^2)} \quad ,$$

respectively.

Returning to (11), we replace x, y by xt, yt, respectively, so that

(23) $$\sum_{n=0}^{\infty} t^n \sum_{r=0}^{n} u_{r,n-r}x^r y^{n-r} = \frac{(x + y)t}{(1 - xt - x^2t^2)(1 - yt - y^2t^2)} \quad .$$

Since the right member of (23) is equal to

$$\frac{x+y}{(x-y)(x^2+3xy+y^2)}\left[\frac{xy+x^2(x+y)t}{1-xt-x^2t^2}-\frac{xy+y^2(x+y)t}{1-yt-y^2t^2}\right]$$

$$=\frac{x+y}{(x-y)(x^2+3xy+y^2}\left\{\left[xy+x^2(x+y)t\right]\sum_{0}^{\infty}u_{n+1}x^n t^n\right.$$

$$\left.-\left[xy+y^2(x+y)t\right]\sum_{0}^{\infty}u_{n+1}y^n t^n\right\},$$

it follows that

$$\sum_{r=0}^{n}u_{r,n-r}x^r y^{n-r}=\frac{xy(x+y)(x^n-y^n)u_{n-1}+(x+y)^2(x^{n+1}-y^{n+1})u_n}{(x-y)(x^2+3xy+y^2)}\,. \tag{24}$$

The polynomials

$$D_n = D_n(x,y) = \sum_{r=0}^{n}u_{r,n-r}\,x^r y^{n-r}$$

correspond to the secondary diagonals in the Fibonacci array. For example, we have

$$D_0 = 0,\qquad D_1 + x+y,\qquad D_2 = (x+y)^2,$$

$$D_3 = 2(x+y)^3 - 3xy(x+y),$$

$$D_4 = 3(x+y)^4 - 7xy(x+y)^2\,.$$

Since

$$\frac{x^{n+1}-y^{n+1}}{x-y}=\sum_{2r\le n}(-1)^r\binom{n-r}{r}(xy)^r(x+y)^{n-2r}\,,$$

we find, after a little manipulation, that (24) implies

(25)
$$D_n(x,y) = -\sum_r \left[\binom{n-r}{r} u_n - \binom{n-r}{r-1} u_{n+1}\right] (x+y)^{n-2r+2}$$

$$\times \frac{(x+y)^{2r} - (-1)^r (xy)^r}{(x+y)^2 + xy} .$$

In particular, if we take

$$x = \alpha = \frac{1+\sqrt{5}}{2}, \quad y = \beta = \frac{1-\sqrt{5}}{2},$$

(25) reduces to

(26)
$$D_n(\alpha, \beta) = \sum_r \left[\binom{n-r}{r-1} u_{n+1} - \binom{n-r}{r} u_n\right] r .$$

However, it is simpler to make use of (11). It is easily verified that

$$\sum_{n=0}^{\infty} D_n(\alpha,\beta)t^n = \frac{t}{(1+t)^2(1-3t+t^2)} = (1+t)^{-2} \sum_{n=0}^{\infty} u_{2n} t^n ,$$

so that

(27)
$$D_n(\alpha,\beta) = \sum_{r=0}^{n} (-1)^r (r+1) u_{2n-2r} .$$

It is not obvious that (26) and (27) are identical. As an instance of (27), we have

$$D_4(\alpha, \beta) = u_8 - 2u_6 + 3u_4 - 4u_2 + 5u_0 = 21 - 16 + 9 - 4 = 10.$$

In the next place we evaluate the determinant

$$\triangle(r,\ s;\ m,\ n) = \begin{vmatrix} u_{r,m} & u_{r,n} \\ u_{s,m} & u_{s,n} \end{vmatrix} .$$

Using (10) we get

$$\triangle(r, s; m, n) = (u_r u_{s+1} - u_{r+1} u_s)(u_{m+1} u_n - u_m u_{n+1}) .$$

Since, for $n \geq m$,

$$u_{m+1}u_n - u_m u_{n+1} = -(u_m u_{n-1} - u_{m-1}u_n) = (-1)^m (u_1 u_{n-m} - u_0 u_{n-m+1})$$

$$= (-1)^m u_{n-m},$$

it follows that

(28) $$\triangle(r, s; m, n) = (-1)^{m+r+1} u_{n-m} u_{s-r} \quad (n \geq m,\ s \geq r).$$

In particular, when $m = r$, $n = s$, (28) becomes

(29) $$\triangle(r, s; r, s) = -u_{s-r}^2 \qquad (s \geq r) .$$

Consider the symmetric matrix of order n:

(30) $$M_n = (u_{r,s}) \qquad (r, s = 0, 1, \ldots, n-1) .$$

Clearly the rank of $M_n \leq 2$ and indeed is equal to 2 for $n \geq 2$. The characteristic polynomial of M_n is given by

$$p_n(x) = x^n - \sum_{r=0}^{n-1} u_{r,r} x^{n-1} + \sum_{0 \leq r < s < n} \triangle(r, s; r, s) x^{n-2}$$

The coefficient of x^{n-1} can be found by means of (17). As for the coefficient of x^{n-2}, it follows from (29) that

$$\sum_{o \le r < s < n} \Delta(r, s; r, s) = -\sum_{o \le r < s < n} u_{s-r}^2 = -\sum_{r=0}^{n-2} \sum_{s=r+1}^{n-1} u_{s-r}^2$$

$$= -\sum_{r=0}^{n-2} \sum_{s=1}^{n-r-1} u_s^2 = -\sum_{r=0}^{n-1} \sum_{s=0}^{n-r-1} u_s^2 .$$

But

$$5 \sum_{s=0}^{n-1} u_s^2 = \sum_{s=0}^{n-1} \left[\alpha^{2s} + \beta^{2s} - 2(-1)^s\right] = \frac{1 - \alpha^{2n}}{1 - \alpha^2} \quad \frac{1 - \beta^{2n}}{1 - \beta^2} - 2\epsilon_n$$

$$= 1 - v_{2n-2} + v_{2n} - 2\epsilon_n ,$$

where as above $v_n = \alpha^n + \beta^n$ and

(31) $$\epsilon_n = \begin{cases} 0 & \text{(n even)} \\ 1 & \text{(n odd)} . \end{cases}$$

Then

$$5 \sum_{r=0}^{n-1} \sum_{s=0}^{n-r-1} u_s^2 = \sum_{r=0}^{n-1} (1 - v_{2n-2r-2} + v_{2n-2r} - 2\epsilon_{n-r})$$

$$= n - 2 + v_{2n} - 2\left[\frac{1}{2}(n + 1)\right] = v_{2n} - 2 - \epsilon_n,$$

so that

(32) $$\sum_{o \le r < s < n} \Delta(r, s; r, s) = -\frac{1}{5}(v_{2n} - 2 - \epsilon_n) .$$

Therefore, using (17) and (32), we find that the characteristic polynomial of M_n is given by

$$(33) \qquad p_n(x) = \begin{cases} x^n - 2u_n^2 x^{n-1} - u_n^2 x^{n-2} & (n \text{ even}) \\ x^n - 2u_{n+1}u_{n-1}x^{n-1} - (u_n^2 - 1)x^{n-2} & (n \text{ odd}, n > 1). \end{cases}$$

For example, we have

$$p_2(x) = x^2 - 2x - 1 \quad , \quad p_3(x) = x^3 - 6x^2 - 3x \quad ,$$

as can be verified directly.

By means of (33) we can compute the characteristic values of M_n. In addition to $n - 2$ zeros we have

$$(34) \qquad \begin{cases} u_n^2 \pm u_n \sqrt{u_n^2 + 1} & (n \text{ even}) \\ u_{n+1} u_{n-1} \pm \sqrt{u_{n+1}^2 u_{n-1}^2 + u_n^2 - 1} & (n \text{ odd}) \; . \end{cases}$$

S. L. Basin, 'Fibonacci Numbers,' presented to the Cupertino High School Mathematics Club, Cupertino, Calif., February 11, 1963.

Brother U. Alfred, 'Fibonacci Discovery,' presented to the California Mathematics Council, Northern Section, at St. Mary's College, Calif., March 30, 1963.

Verner E. Hoggatt, Jr., 'Fibonacci Numbers,' presented to the California Mathematics Council, Northern Section, at St. Mary's College, Calif., March 30, 1963.

D. W. Robinson, 'The Fibonacci Matrix Modulo m,' presented to the Mathematical Association of America, March 9, 1963.

H. W. Gould and T. A. Chapman, 'Solution of Functional Equations Involving Turán Expressions,' presented to the West Virginia Academy of Science, April 26, 1963.

H. W. Gould, 'A b-parameter Series Transform with Novel Applications to Bessel and Legendre polynomials,' presented to the American Mathematical Association, May 4, 1963.

N. J. Fine, 'An Elementary Arithmetic Measure,' presented to Allegheny Mountain Section of the American Mathematical Association, May 4, 1963.

FIBONACCI RELATED MASTER'S THESES

1. John E. Vinson, 'Modulo m properties of the Fibonacci Sequence,' Oregon State University, 1961, Advisor: Prof. Robert Stalley.
2. Charles H. King, 'Some Properties of the Fibonacci Sequence,' San Jose State College, 1960, Advisor: Prof. Verner E. Hoggatt, Jr.
3. Richard A. Hayes, 'Fibonacci and Lucas Polynomials,' San Jose State College, Advisor: Prof. Verner E. Hoggatt, Jr. (Not yet completed.)
4. Sister Mary de Sales McNabb, 'Fibonacci Numbers: Some Properties and Generalizations' Catholic University of America, Advisor: Prof. Raymond W. Moller. (Not yet completed.)

FIBONACCI ARTICLES SOON TO APPEAR

S. L. Basin, An Application of Continuants as a Link Between Chebyshev and Fibonacci, Mathematics Magazine.

D. Zeitlin, On Identities for Fibonacci Numbers, Classroom Notes, American Mathematical Monthly.

A. F. Horadam, On Khazanov's Formulae, Mathematics Magazine.

D. E. Thoro, Regula Falsi and the Fibonacci Numbers, The American Mathematical Monthly.