GENERALIZED FIBONACCI POLYNOMIALS AND ZECKENDORF'S THEOREM

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1. INTRODUCTION

The Zeckendorf Theorem states that every positive integer can be uniquely represented as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are used in any given sum. In fact, in an earlier paper [1], it was shown that every positive integer can be uniquely represented as the sum from k copies of distinct members of the generalized Fibonacci sequence formed by evaluating the Fibonacci polynomials at $x = k$, if no two consecutive members of the sequence with coefficient k are used in any given sum. Now, this result is extended to include sequences formed from generalized Fibonacci polynomials evaluated at $x = k$.

2. THE GENERALIZED ZECKENDORF THEOREM FOR THE TRIBONACCI POLYNOMIALS

The Tribonacci polynomials have been defined in [2] as $T_{-1}(x) = T_0(x) = 0$, $T_1(x) = 1$, $T_{n+3}(x) = x^2T_{n+2}(x) + xT_{n+1}(x) + T_n(x)$. Let us say that the Tribonacci polynomials are evaluated at $x = k$. Then the number sequence is $U_{-1} = U_0 = 0$, $U_1 = 1$, $U_2 = k^2$,

$$U_{n+3} = k^2U_{n+2} + kU_{n+1} + U_n .$$

Theorem 2.1. Let U_n be the n^{th} member of the sequence formed when the Tribonacci polynomials are evaluated at $x = k$. Then every positive integer N has a unique representation in the form

$$N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$$

with the constraints

$$\epsilon_1 = 0, 1, 2, \cdots, k^2 - 1,$$

and, for $i \geq 2$,

$$\epsilon_i = 0, 1, 2, \cdots, k^2 ;$$

If $\epsilon_2 = k^2$, then $\epsilon_1 = 0, 1, \cdots, k - 1$;
If $\epsilon_{i+1} = k^2$, then $\epsilon_i = 0, 1, 2, \cdots, k$;
If $\epsilon_{i+1} = k^2$ and $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

We begin with three useful lemmas, which can be proved by mathematical induction or by considering how to represent the specific integers given using the constraints of Theorem 2.1.

Lemma 1. For $k \geq 2$,

$$\begin{aligned} U_{3n} - 1 &= k^2(U_{3n-1} + U_{3n-4} + \cdots + U_2) + k(U_{3n-2} + U_{3n-5} + \cdots + U_1) - 1 \\ &= k^2(U_{3n-1} + \cdots + U_2) + k(U_{3n-2} + U_{3n-5} + \cdots + U_4) + (k-1)U_1 . \end{aligned}$$

Lemma 2. For $k \geq 2$,

$$U_{3n+1} - 1 = k^2(U_{3n} + U_{3n-3} + \cdots + U_3) + k(U_{3n-1} + U_{3n-4} + \cdots + U_2).$$

Lemma 3. For $k \geq 2$,

$$U_{3n+2} - 1 = k^2(U_{3n+1} + U_{3n-2} + \cdots + U_4) + k(U_{3n} + U_{3n-3} + \cdots + U_3) + (k^2 - 1)U_1.$$

These lemmas are almost self-explanatory. Now for the utility of the three lemmas in the proof of Theorem 2.1.

Assume that every integer $s \leq U_{3n+2} - 1$ has a unique admissible representation. By using rU_{3n+2} for $r = 1, 2, \cdots, k^2$, one can get a representation for $s \leq (k^2 - 1)U_{3n+2} + (U_{3n+2} - 1)$ without using k^2U_{3n+2}. If we now add another U_{3n+2}, then k^2U_{3n+2} is representable but now the representation for $U_{3n+2} - 1$ cannot be used since U_{3n+1} has too large a coefficient. Let $k^2U_{3n+2} \geq s$ be representable in admissible form. Now we gradually build up to $(k - 1)U_{3n+1}$, and since $k^2 = \epsilon_{i+1}$ and $\epsilon_i = k - 1$, so that there are no restrictions on the earlier coefficients because we can still obtain $U_{3n+1} - 1$ without further conflict.

Thus $s \leq k^2U_{3n+2} + (k - 1)U_{3n+1} + U_{3n+1} - 1 = k^2U_{3n+2} + kU_{3n+1} - 1$ is obtainable. We cannot now add another U_{3n+1} since $\epsilon_{i+1} = k^2$, $\epsilon_i = k$, so that $\epsilon_{i-1} = 0$ and we cannot now use the representation of $U_{3n+1} - 1$, but we now achieve, without U_{3n}, the sum as great as $U_{3n} - 1$. Now we have a representation up to $s \leq k^2U_{3n+2} + kU_{3n+1} + U_{3n} - 1 = U_{3n+3} - 1$, which completes the proof of Theorem 2.1 by mathematical induction.

3. HIERARCHY OF RESULTS: ZECKENDORF'S THEOREM FOR THE GENERALIZED FIBONACCI POLYNOMIAL SEQUENCES

Define the generalized Fibonacci polynomials as in [2] by

$$P_{-(r-2)}(x) = P_{-(r-3)}(x) = \cdots = P_{-1}(x) = P_0(x) = 0, \quad P_1(x) = 1, \quad P_2(x) = x^{r-1},$$
$$P_{n+r}(x) = x^{r-1}P_{n+r-1}(x) + x^{r-2}P_{n+r-2}(x) + \cdots + P_n(x).$$

Let $U_n = P_n(k)$, the n^{th} member of the sequence formed by evaluating the generalized Fibonacci polynomials at $x = k$. We state the Zeckendorf Theorem for selected values of r.

Theorem 3.1. The Binary Case, $r = 1$. Let $P_n(x) = x^{n-1}$, or, $P_0(x) = 0$, $P_1(x) = 1$, $P_2(x) = x$, $P_{n+1}(x) = xP_n(x)$. Now, if $U_n = P_n(k) = k^{n-1}$, then any positive integer N has a unique representation in the form

$$N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$$

if and only if $\epsilon_i = 0, 1, 2, \cdots, k - 1$.

Theorem 3.1 provides, for example, the representation of a number in decimal notation. Theorem 3.2 is the generalized Zeckendorf Theorem proved in [2], and Theorem 3.3 is Theorem 2.1 restated.

Theorem 3.2. The Fibonacci Case, $r = 2$. Let $P_{n+2}(x) = xP_{n+1}(x) + P_n(x)$, $P_0(x) = 0$, $P_1(x) = 1$. Let $U_n = P_n(k)$. Then every positive integer N has a unique representation in the form $N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$ if and only if $\epsilon_1 = 0, 1, 2, \cdots, k - 1$, and for $i \geq 2$,

$$\epsilon_i = 0, 1, 2, \cdots, k;$$

If $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

Theorem 3.3. The Tribonacci Case, $r = 3$. Let $P_{-1}(x) = P_0(x) = 0$, $P_1(x) = 1$, and $P_{n+3}(x) = x^2P_{n+2}(x) + xP_{n+1}(x) + P_n(x)$, and let $U_n = P_n(k)$. Then every positive integer N has a unique representation in the form $N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$ if and only if

$$\epsilon_1 = 0, 1, 2, \cdots, k^2 - 1$$

and for $i \geq 2$,

$$\epsilon_i = 0, 1, 2, \cdots, k^2;$$

If $\epsilon_2 = k^2$, then $\epsilon_1 = 0, 1, \cdots, k - 1$;

If $\epsilon_{i+1} = k^2$, then $\epsilon_i = 0, 1, 2, \cdots, k$;

If $\epsilon_{i+1} = k^2$ and $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

Theorem 3.4. The Quadranacci Case, $r = 4$. Let $P_{-2}(x) = P_{-1}(x) = P_0(x) = 0$, $P_1(x) = 1$, $P_2(x) = x^3$, $P_{n+4}(x) = x^3P_{n+3}(x) + x^2P_{n+2}(x) + xP_{n+1}(x) + P_n(x)$. Let $U_n = P_n(k)$, $k \geq 1$. Then any positive integer N has a unique representation in the form

$$N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$$

if

$$\epsilon_1 = 0, 1, 2, \cdots, k^3 - 1,$$

and, for $i \geq 2$,

$$\epsilon_i = 0, 1, 2, \cdots, k^3;$$

If $\epsilon_2 = k^3$, then $\epsilon_1 = 0, 1, 2, \cdots, k^2 - 1$;

If $\epsilon_2 = k^3$ and $\epsilon_2 = k^2$, then $\epsilon_1 = 0, 1, 2, \cdots, k - 1$;

If $\epsilon_{i+2} = k^3$, then $\epsilon_{i+1} = 0, 1, 2, \cdots, k^2$;

If $\epsilon_{i+2} = k^3$ and $\epsilon_{i+1} = k^2$, then $\epsilon_i = 0, 1, 2, \cdots, k$;

If $\epsilon_{i+2} = k^3$, $\epsilon_{i+1} = k^2$, and $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

Theorem 3.5. The Pentanacci Case, $r = 5$. Let $P_{-3}(x) = P_{-2}(x) = P_{-1}(x) = P_0(x) = 0$, $P_1(x) = 1$, $P_2(x) = x^4$, and

$$P_{n+5}(x) = x^4P_{n+4}(x) + x^3P_{n+3}(x) + x^2P_{n+2}(x) + xP_{n+1}(x) + P_n(x),$$

and then let $U_n = P_n(k)$. Then every positive integer N can be represented uniquely in the form

$$N = \epsilon_1U_1 + \epsilon_2U_2 + \cdots + \epsilon_nU_n$$

if

$$\epsilon_1 = 0, 1, 2, \cdots, k^4 - 1$$

and, for $i \geq 2$, $\epsilon_i = 0, 1, 2, \cdots, k^4$

where

If $\epsilon_2 = k^4$, then $\epsilon_1 = 0, 1, 2, \cdots, k^3 - 1$;

If $\epsilon_3 = k^4$ and $\epsilon_2 = k^3$, then $\epsilon_1 = 0, 1, \cdots, k^2 - 1$;

If $\epsilon_4 = k^4$ and $\epsilon_3 = k^3$ and $\epsilon_2 = k^2$, then $\epsilon_1 = 0, 1, 2, \cdots, k - 1$;

If $\epsilon_{i+3} = k^4$, then $\epsilon_{i+2} = 0, 1, \cdots, k^3$;

If $\epsilon_{i+3} = k^4$ and $\epsilon_{i+2} = k^3$, then $\epsilon_{i+1} = 0, 1, 2, \cdots, k^2$;

If $\epsilon_{i+3} = k^4$, $\epsilon_{i+2} = k^3$, and $\epsilon_{i+1} = k^2$, then $\epsilon_i = 0, 1, \cdots, k$;
If $\epsilon_{i+3} = k^4$, $\epsilon_{i+2} = k^3$, $\epsilon_{i+1} = k^2$, and $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

A proof by mathematical induction of Theorem 3.5 requires five lemmas given below.

Lemma 1.

$$\begin{aligned} U_{5n} - 1 = {} & k^4U_{5n-1} + k^3U_{5n-2} + k^2U_{5n-3} + kU_{5n-4} \\ & + k^4U_{5n-6} + k^3U_{5n-7} + k^2U_{5n-8} + kU_{5n-9} \\ & + \cdots + \cdots + \cdots + \cdots \\ & + k^4U_4 + k^3U_3 + k^2U_2 + (k-1)U_1 . \end{aligned}$$

Lemma 2.

$$\begin{aligned} U_{5n+1} - 1 = {} & k^4U_{5n} + k^3U_{5n-1} + k^2U_{5n-2} + kU_{5n-3} \\ & + k^4U_{5n-5} + k^3U_{5n-6} + k^2U_{5n-8} + kU_{5n-9} \\ & + \cdots + \cdots + \cdots + \cdots \\ & + k^4U_5 + k^3U_4 + k^2U_3 + kU_2 + 0(U_1 - 1) . \end{aligned}$$

Lemma 3.

$$\begin{aligned} U_{5n+2} - 1 = {} & k^4U_{5n+1} + k^3U_{5n} + k^2U_{5n-1} + kU_{5n-2} \\ & + k^4U_{5n-4} + k^3U_{5n-5} + k^2U_{5n-6} + kU_{5n-7} \\ & + \cdots + \cdots + \cdots + \cdots \\ & + k^4U_6 + k^3U_5 + k^2U_4 + kU_3 + (k^4 - 1)U_1 . \end{aligned}$$

Lemma 4.

$$\begin{aligned} U_{5n+3} - 1 = {} & k^4U_{5n+2} + k^3U_{5n+1} + k^2U_{5n} + kU_{5n-1} \\ & + k^4U_{5n-3} + k^3U_{5n-4} + k^2U_{5n-5} + kU_{5n-6} \\ & + \cdots + \cdots + \cdots + \cdots \\ & + k^4U_7 + k^3U_6 + k^2U_5 + kU_4 \\ & + k^4U_2 + (k^3 - 1)U_1 . \end{aligned}$$

Lemma 5.

$$\begin{aligned} U_{5n+4} - 1 = {} & k^4U_{5n+3} + k^3U_{5n+2} + k^2U_{5n+1} + kU_{5n} \\ & + k^4U_{5n-1} + k^3U_{5n-2} + k^2U_{5n-3} + kU_{5n-4} \\ & + \cdots + \cdots + \cdots + \cdots \\ & + k^4U_8 + k^3U_7 + k^2U_6 + kU_5 \\ & + k^4U_3 + k^3U_2 + (k^2 - 1)U_1 . \end{aligned}$$

Theorem 3.6. Let $P_{-(r-2)}(x) = P_{-(r-3)}(x) = \cdots = P_{-1}(x) = P_0(x) = 0$, $P_1(x) = 1$, $P_2(x) = x^{r-1}$, and $P_{n+r}(x) = x^{r-1}P_{n+r-1}(x) + x^{r-2}P_{n+r-2}(x) + \cdots + P_n(x)$. and then let $U_n = P_n(k)$. Then every positive integer N has a unique representation in the form

$$N = \epsilon_1 U_1 + \epsilon_2 U_2 + \cdots + \epsilon_n U_n$$

if

$$\epsilon_1 = 0, 1, 2, \cdots, k^{r-1} - 1$$

and, for $i \geq 2$,

$$\epsilon_i = 0, 1, 2, \cdots, k^{r-1},$$

where

If $\epsilon_2 = k^{r-1}$, then $\epsilon_1 = 0, 1, 2, \cdots, k^{r-2} - 1$;

If $\epsilon_3 = k^{r-1}$ and $\epsilon_2 = k^{r-2}$, then $\epsilon_1 = 0, 1, 2, \cdots, k^{r-3} - 1$;

$\cdots$

If $\epsilon_{r-1} = k^{r-1}$, $\epsilon_{r-2} = k^{r-2}$, $\cdots$, and $\epsilon_2 = k^2$, then $\epsilon_1 = 0, 1, 2, \cdots, k - 1$;

If $\epsilon_{i+r-2} = k^{r-1}$, then $\epsilon_{i+r-3} = 0, 1, \cdots, k^{r-2}$;

If $\epsilon_{i+r-2} = k^{r-1}$ and $\epsilon_{i+r-3} = k^{r-2}$, then $\epsilon_{i+r-4} = 0, 1, \cdots, k^{r-3}$;

$\cdots$

If $\epsilon_{i+r-2} = k^{r-1}$, $\epsilon_{i+r-3} = k^{r-2}$, $\cdots$, $\epsilon_{i+1} = k^2$, then $\epsilon_i = 0, 1, \cdots, k$;

If $\epsilon_{i+r-2} = k^{r-1}$, $\epsilon_{i+r-3} = k^{r-2}$, $\cdots$, $\epsilon_{i+1} = k^2$, $\epsilon_i = k$, then $\epsilon_{i=1} = 0$.

The number of conditions increases, of course, as r increases. For $r = 2$, the Fibonacci case needs 3 constraints; for $r = 3$, 5 constraints; for $r = 4$, 7 constraints; for $r = 5$, 9 constraints, and for the r^{th} case, $2r - 1$ constraints are needed and r identities must be used in the inductive proof.

4. THE ZECKENDORF THEOREM FOR SIMULTANEOUS REPRESENTATIONS

Klarner has proved the following theorem in [3]:

Klarner's Theorem. Given non-negative integers A and B, there exists a unique set of integers $\{k_1, k_2, k_3, \cdots, k_r\}$ such that

$$A = F_{k_1} + F_{k_2} + \cdots + F_{k_r},$$
$$B = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1},$$

for $|k_i - k_j| \geq 2$, $i \neq j$, where each F_i is an element of the sequence $\{F_n\}_{-\infty}^{+\infty}$, the double-ended Fibonacci sequence, $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$.

This is a new Zeckendorf theorem for simultaneous representation. Actually, integers A and B are so representable if and only if

$$\alpha B + A \geq 0, \quad 1 < \alpha < 2, \quad \alpha^2 = \alpha + 1, \quad \alpha = (1 + \sqrt{5})/2,$$

with equality if and only if $A = B = 0$, the vacuous representation of $(0,0)$ using no representing Fibonacci numbers.

From the fact that $\alpha B + A \geq 0$ is a condition for representability, it follows that every integer can be either an A or a B and can have a proper representation. For instance,

$$-100 = F_{k_1} + F_{k_2} + \cdots + F_{k_r}$$

for some $\{k_1, k_2, \cdots, k_r\}$, $|k_i - k_j| \geq 2$, $i \neq j$. The line $x = -100$ cuts the line $\alpha x + y = 0$, say, in $(-100, y_0)$. Then let $y_1 > y_0$ be an integer, and $-100\alpha + y_1 > 0$, and so -100 has a representation, and indeed has an infinite number of such representations as all integers $y_k > y_1$ give rise to admissible representations.

Now, given positive integers A and B, $B > A$, how does one find the simultaneous representation of Klarner's Theorem? To begin, write the Zeckendorf minimal representation for $A + B$,

$$A + B = F_{m_1} + F_{m_2} + \cdots + F_{m_n} ,$$

where $|m_i - m_j| \geq 2$, $i \neq j$, $m_1 > m_2 > m_3 > \cdots > m_n$. Then $B = F_{m_1-1} + R_B$ and $A = F_{m_1-2} + R_A$. The next Fibonacci numbers in the representations of B and A are F_{m_2-1} and F_{m_2-2} if $R_B \geq R_A \geq 0$, not both R_B and $R_A = 0$. When m_2 is odd and $R_B < R_A$ or $R_A < 0$, then the next Fibonacci numbers in the representations of B and A are F_{-m_2-1} and F_{-m_2-2}. The process continues, so that the sums of successive terms in the representations of A and of B give the successive terms in $A + B$, except that the last terms may have a zero sum, and the subscripts in the representations of A and B may not be ordered.

We give a constructive proof of Klarner's Theorem using mathematical induction. First, $A = 0$ and $B = 1$ is given uniquely by $A = F_0$ and $B = F_1$, while $A = 1$ and $B = 0$ is given uniquely by $A = F_{-1}$ and $B = F_0$. Here, of course, we seek minimal representations in the form

$$A = F_{k_1} + F_{k_2} + \cdots + F_{k_r} ,$$
$$B = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1} ,$$

for $F_i \in \{F_n\}_{-\infty}^{+\infty}$ with $|k_i - k_j| \geq 2$, $i \neq j$ (the conditions for the original Zeckendorf Theorem), and, of course, we assume that

$$k_1 < k_2 < k_3 < \cdots < k_r .$$

If we make the inductive assumption that all integers $A \geq 0$, $B \geq 0$, $0 \leq A + B \leq n$ can be so represented, then we must secure compatible pairs $A + 1, B$ and $A, B + 1$ each in admissible form from those of the pair A, B. We do this as follows. Let

$$A + 1 = F_{k_1} + \cdots + F_{k_r} + F_{-1} ,$$
$$B = F_{k_1+1} + \cdots + F_{k_r+1} + F_0 ;$$

then we must put $A + 1$ and B into admissible form. We will show how to put these into admissible form in general by putting

$$F_{k_1} + F_{k_2} + \cdots + F_{k_r} + F_m$$

into admissible form. Now, if F_m is detached, there is no problem. If F_m and F_{k_j} are adjacent, then simply use the formula $F_{v+1} = F_v + F_{v-1}$ to work upward in the subscripts until the stress is relieved. Since we have only a finite number r, there is no problem.

Now, if $F_m = F_{k_j}$, then $2F_m = F_m + F_{m-1} + F_{m-2} = F_{m+1} + F_{m-2}$. This may cause $2F_{m-2}$ and also the condition $F_{m+2} + F_{m+1}$. If the latter, use $F_{v+1} = F_v + F_{v-1}$ to move upward in the subscripts to relieve the stress. We notice now that the $2F_{m-2}$ has a smaller subscript than before. Repeat the process. This ultimately terminates or forms two consecutive Fibonacci numbers, where we can use $F_{v+1} = F_v + F_{v-1}$ to relieve the stress, and we are done.

Notice that this same procedure on B leaves the relation between A + 1 and B intact. You can also consider $F_m = F_{-1}$ or $F_m = 0$, etc.

Next, form the sequence $G_0 = A$, $G_1 = B$, $G_{n+2} = G_{n+1} + G_n$, and assume that A (and hence B) has two distinct admissible forms. Then let n become so large that all Fibonacci subscripts are positive, and we will violate the original Zeckendorf Theorem, for G_n would have two distinct representations. Then, A and B must have unique representations in the admissible form.

But, all of this is extendable. The double-ended Lucas sequence, $\{L_n\}_{-\infty}^{+\infty}$, $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$, also enjoys the representation property of A and B, for $\alpha B + A \geq 0$, except that, additionally, A and B are chosen such that $5 \,|\, (A^2 + B^2, A^2 + 2AB)$, so that not every integer pair qualifies.

We can generalize Klarner's Theorem to apply to the sequences formed when the Fibonacci polynomials are evaluated at $x = k$ as follows.

<u>Theorem 4.1.</u> Given non-negative integers A and B, there exists a unique set of integers, $\{\epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_r, j\}$ such that

$$A = \epsilon_1 U_{j+1} + \epsilon_2 U_{j+2} + \cdots + \epsilon_r U_{j+r},$$
$$B = \epsilon_1 U_{j+2} + \epsilon_2 U_{j+3} + \cdots + \epsilon_r U_{j+r+1},$$

where each U_i is an element of the sequence $\{U_n\}_{-\infty}^{+\infty}$, the double-ended sequence formed from the Fibonacci polynomials, $F_0(x) = 0$, $F_1(x) = 1$, $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, when $x = k$, so that $U_n = F_n(k)$, and the ϵ_i satisfy the constraints $\epsilon_i = 0, 1, 2, \cdots, k$, and if $\epsilon_i = k$, then $\epsilon_{i-1} = 0$, $-\infty < i < +\infty$.

<u>Proof.</u> When $k = 1$, we have Klarner's Theorem. We take $k \geq 2$. First, we can represent uniquely $A = 0 = U_0$ and $B = 1 = U_1$ or $A = 1 = U_{-1}$ and $B = 0 = U_0$. If we make the inductive assumption that all integers $A \geq 0$, $B \geq 0$, $0 \leq A + B \leq n$ can be so represented, then we must secure compatible pairs A + 1, B and A, B + 1 each in admissible form from the pair A, B. Let

$$A + 1 = \epsilon_1 U_{j+1} + \epsilon_2 U_{j+2} + \cdots + \epsilon_r U_{j+r} + U_{-1},$$
$$B = \epsilon_1 U_{j+2} + \epsilon_2 U_{j+3} + \cdots + \epsilon_r U_{j+r+1} + U_0 .$$

We show how to put these into admissible form by working with

$$\epsilon_1 U_{j+1} + \epsilon_2 U_{j+2} + \cdots + \epsilon_r U_{j+r} + U_m .$$

Case 1. U_m is away from any other U_{j+i} or no $\epsilon_i = k$. We are done; there is no interference unless we create $\epsilon_i + 1 = k$, in which case we may have to use $U_{j+1} = kU_j + U_{j-1}$.

Case 2. $\epsilon_i = k$ and $U_m = U_{j+i-1}$; then replace $kU_j + U_{j-1}$ by U_{j+1} and work the subscripts upward to relieve the stress. Notice that this process always terminates, thus minimizing the number of terms. Suppose that $U_m = U_{j+i}$ with $\epsilon_i = k$; then

$$\begin{aligned}(k + 1)U_{j+i} &= (U_{j+i+1} - U_{j+i-1}) + kU_{j+i-1} + U_{j+i-2}\\ &= U_{j+i+1} + (k - 1)U_{j+i-1} + U_{j+i-2}\,.\end{aligned}$$

Now $\epsilon_{j+1} \neq k$ (since $\epsilon_j = k$), so, if $\epsilon_{j+2} = k$, then $kU_t + U_{t-1} = U_{t+1}$ can be used and the subscripts can be worked upwards to relieve the stress. If the coefficient of U_{j+i-2} is now $(k + 1)$, we note that we can repeat the process and ultimately work it out downward while using $kU_t + U_{t-1} = U_{t+1}$ to relieve the stress upward. In any case, this algorithm will reduce $A + 1$ to an admissible form. Thus, we can represent A' and B' for $0 \leq A' + B' \leq n + 1$, finishing a proof that pairs of integers are so representable.

Next, to show uniqueness, form the sequence $V_0 = A$, $V_1 = B$, $V_{n+2} = kV_{n+1} + V_n$, and assume that A, and hence B, has two distinct admissible forms. Then, V_n has two distinct representations from those of, say A. Then, if n is large enough, V_n must have terms $\epsilon_i U_i$ which all have positive subscripts, and V_n has two distinct representations, which violates the generalized Zeckendorf Theorem for the Fibonacci polynomials evaluated at $x = k$ given in [1], which guarantees a unique representation.

The condition for representability for the Fibonacci polynomials evaluated at $x = k$ is

$$B(k + \sqrt{k^2 + 4})/2 + A \geq 0$$

with equality only if $A = B = 0$ is vacuously represented.

Now, if we use the Tribonacci numbers, 1, 1, 2, $\cdots$, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, then we can represent any three non-negative integers A, B, and C as in Klarner's Theorem. For the Tribonacci numbers,

$$C\lambda^2 + B(\lambda + 1) + A \geq 0, \quad 1 < \lambda < 2, \quad \lambda^3 = \lambda^2 + \lambda + 1\,,$$

is the condition for the ordered triple A, B, C to be a lattice point in the representation half-space, and

$$x\lambda^2 + y(\lambda + 1) + z = 0$$

is the separator plane containing $(0, 0, 0)$ to be represented vacuously using no Tribonacci numbers. We now generalize Klarner's Theorem to Tribonacci numbers.

Theorem 4.2. Given three non-negative integers A, B, and C, there exists a unique set of integers $\{k_1, k_2, \cdots, k_r\}$ such that

$$A = T_{k_1} + T_{k_2} + \cdots + T_{k_r}\,,$$

$$B = T_{k_1+1} + T_{k_2+1} + \cdots + T_{k_r+1} \quad ,$$
$$C = T_{k_1+2} + T_{k_2+2} + \cdots + T_{k_r+2} \quad ,$$

where $k_1 < k_2 < k_3 < \cdots < k_r$ and no three k_i, k_{i+1}, k_{i+2} are consecutive integers, and where the T_i are members of the sequence $\{T_n\}_{-\infty}^{+\infty}$, the double-ended sequence of Tribonacci numbers, $T_{-1} = T_0 = 0$, $T_1 = 1$, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$.

Proof. First, $A = 1 = T_{-2}$, $B = 0 = T_{-1}$, $C = 0 = T_0$; and $A = 0 = T_{-3} + T_{-2}$, $B = 1 = T_{-2} + T_{-1}$, $C = 0 = T_{-1} + T_0$; and $A = 0 = T_{-1}$, $B = 0 = T_0$, $C = 1 = T_1$ are given uniquely. We make the inductive assumption that all integers $A \ge 0$, $B \ge 0$, $C \ge 0$, $0 \le A + B + C \le n$ can be represented uniquely in the form of the theorem. We must show that we can secure the compatible triples $A + 1, B, C$; $A, B + 1, C$; and $A, B, C + 1$ in admissible form from the triple A, B, C to get the representations for $(A + 1) + B + C \le n + 1$, $A + (B + 1) + C \le n + 1$, $A + B + (C + 1) \le n + 1$. To get $A + 1, B, C$, we add T_{-2} to A, T_{-1} to B, and T_0 to C, and then work upwards in the subscripts if necessary. To get $A, B + 1, C$ from A, B, and C, we add $T_{-3} + T_{-2}$ to the representation for A, $T_{-2} + T_{-1}$ to B, and $T_{-1} + T_0$ to C. To get $A, B, C + 1$ from the representations for A, B, and C, we add respectively T_{-1}, T_0, T_1. Thus, given the representations for A, B, C, $0 \le A + B + C \le n$, we can always make the representation for one member of the triple to be increased by 1, so that we can represent all numbers whose sum is less than or equal to $n + 1$.

Uniqueness follows from Theorem 3.3 with $k = 1$.

Theorem 4.2 can be generalized to the general Tribonacci numbers obtained when the Tribonacci polynomials are evaluated at $x = k$. In that proof, one would obtain $A + 1$, B, C from A, B, C by adding T_{-2}, T_{-1}, and T_0 to A, B, and C, respectively; $A, B + 1, C$ by adding $T_{-3} + kT_{-2}$ to A, $T_{-2} + kT_{-1}$ to B, and $T_{-1} + kT_0$ to C; and $A, B, C + 1$ by adding T_{-1}, T_0, T_1 to A, B, and C, respectively. Theorem 4.3 contains the generalization, as

Theorem 4.3. Given three non-negative integers A, B, and C, there exists a unique set of integers $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_r, j\}$ such that

$$A = \epsilon_1 U_{j+1} + \epsilon_2 U_{j+2} + \cdots + \epsilon_r U_{j+r} \; ,$$
$$B = \epsilon_1 U_{j+2} + \epsilon_2 U_{j+3} + \cdots + \epsilon_r U_{j+r+1} \; ,$$
$$C = \epsilon_1 U_{j+3} + \epsilon_2 U_{j+4} + \cdots + \epsilon_r U_{j+r+2} \; ,$$

where each U_i is an element of the sequence $\{U_n\}_{-\infty}^{+\infty}$, the double-ended sequence given by $U_n = T_n(k)$, $T_{-1}(x) = T_0(x) = 0$, $T_1(x) = 1$, $T_{n+3}(x) = x^2T_{n+2}(x) + xT_{n+1}(x) + T_n(x)$, where the ϵ_i satisfy the constraints $\epsilon_i = 0, 1, 2, \cdots, k^2$, and if $\epsilon_i = k^2$, then $\epsilon_{i-1} = 0, 1, \cdots, k^2 - 1$, and if $\epsilon_i = k^2$ and $\epsilon_{i-1} = k$, then $\epsilon_{i-2} = 0$.

Finally, we can generalize Klarner's Theorem to apply to the sequences which arise when the generalized Fibonacci polynomials are evaluated at $x = k$.

Theorem 4.4. Given r non-negative integers $N_1, N_2, \cdots, N_r$, there exists a unique set of integers $\{k_1, k_2, \cdots, k_s\}$ such that

$$N_i = U_{k_1+i-1} + U_{k_2+i-1} + \cdots + U_{k_s+i-1} ,$$

where $k_1 < k_2 < k_3 < \cdots < k_s$ and no r $k_i, k_{i+1}, \cdots, k_{i+r-1}$ are consecutive integers, where the U_i are members of the sequence $\{U_n\}_{-\infty}^{+\infty}$, the double-ended sequence of r-nacci numbers, $U_{-r} = -1$, $1 = U_{-r+1}$, $U_{-r+2} = U_{-r+3} = \cdots = U_{-1} = U_0 = 0$, $U_1 = 1$, $U_{n+r} = U_{n+r-1} + U_{n+r-2} + \cdots + U_{n+1} + U_n$.

Clearly, an inductive proof of Theorem 4.3, or of the theorem when generalized to the generalized Fibonacci polynomials evaluated at $x = k$, hinges upon being able to add one to one number represented and to again have all r numbers in admissible form. We examine the additions necessary for the inductive step for the generalized Fibonacci polynomials $P_n(x)$ for some small values of r. The induction has been done for $r = 2$ and $r = 3$. For $r = 4$, the Quadranacci polynomials evaluated at $x = k$, where admissible forms are known for A, B, C and D, the additions before adjustment of subscripts are as follows:

$$\begin{aligned} A + 1 &= A + U_{-3} , \\ B &= B + U_{-2} , \\ C &= C + U_{-1} , \\ D &= D + U_0 \quad ; \end{aligned}$$

$$\begin{aligned} A &= A + U_{-4} + kU_{-3} , \\ B + 1 &= B + U_{-3} + kU_{-2} , \\ C &= C + U_{-2} + kU_{-1} , \\ D &= D + U_{-1} + kU_0 \quad ; \end{aligned}$$

$$\begin{aligned} A &= A + U_{-5} + kU_{-4} + k^2U_{-3} , \\ B &= B + U_{-4} + kU_{-3} + k^2U_{-2} , \\ C + 1 &= C + U_{-3} + kU_{-2} + k^2U_{-1} , \\ D &= D + U_{-2} + kU_{-1} + k^2U_0 \quad ; \end{aligned}$$

$$\begin{aligned} A &= A + U_{-2} , \\ B &= B + U_{-1} , \\ C &= C + U_0 \quad , \\ D + 1 &= D + U_1 \quad . \end{aligned}$$

Note that the Quadranacci polynomials extend to negative subscripts as $U_1 = 1$, $U_0 = U_{-1} = U_{-2} = 0$, $U_{-3} = 1$, $U_{-4} = -k$, $U_{-5} = U_{-6} = 0, \cdots$ when evaluated at $x = k$.

For $r = 5$, the Pentanacci polynomials evaluated at $x = k$ are $U_1 = 1$, $U_0 = U_{-1} = U_{-2} = U_{-3} = 0$, $U_{-4} = 1$, $U_{-5} = -k$, $U_{-6} = U_{-7} = 0$, $U_{-8} = 0$ using the relation $U_n =$

$-kU_{n+1} - k^2U_{n+2} - k^3U_{n+3} - k^4U_{n+4} + U_{n+5}$ to move to values for the negative subscripts. The inductive one step pieces for the Pentanacci case, where representations for A, B, C, D and E are given, are

$$\begin{aligned} A + 1 &= A + U_{-4} & A &= A + U_{-5} + kU_{-4} \\ B &= B + U_{-3} & B + 1 &= B + U_{-4} + kU_{-3} \\ C &= C + U_{-2} & C &= C + U_{-3} + kU_{-2} \\ D &= D + U_{-1} & D &= D + U_{-2} + kU_{-1} \\ E &= E + U_{0} & E &= E + U_{-1} + kU_{0} \end{aligned}$$

$$\begin{aligned} A &= A + U_{-6} + kU_{-5} + k^2U_{-4}\,, \\ B &= B + U_{-5} + kU_{-4} + k^2U_{-3}\,, \\ C + 1 &= C + U_{-4} + kU_{-3} + k^2U_{-2}\,, \\ D &= D + U_{-3} + kU_{-2} + k^2U_{-1}\,, \\ E &= E + U_{-2} + kU_{-1} + k^2U_{0}\;; \end{aligned}$$

$$\begin{aligned} A &= A + U_{-7} + kU_{-6} + k^2U_{-5} + k^3U_{-4}\,, \\ B &= B + U_{-6} + kU_{-5} + k^2U_{-4} + k^3U_{-3}\,, \\ C &= C + U_{-5} + kU_{-4} + k^2U_{-3} + k^3U_{-2}\,, \\ D + 1 &= D + U_{-4} + kU_{-3} + k^2U_{-2} + k^3U_{-1}\,, \\ E &= E + U_{-3} + kU_{-2} + k^2U_{-1} + k^3U_{0}\;; \end{aligned}$$

$$\begin{aligned} A &= A + U_{-3}\,, \\ B &= B + U_{-2}\,, \\ C &= C + U_{-1}\,, \\ D &= D + U_{0}\,, \\ E + 1 &= E + U_{1}\,. \end{aligned}$$

Thus the pattern from the first cases is clear. The recurrence relation backward for U_n for general r is

$$U_n = -kU_{n+1} - k^2U_{n+2} - k^3U_{n+3} - \cdots - k^{r-1}U_{n+r-1} + U_{n+r}\,,$$

which leads to the lemma for general r and $k \geq 2$;

<u>Lemma.</u> $U_{-(2r-2)} = \cdots = U_{-(r+1)} = 0$ ($r - 2$ zeroes), $U_{-r} = -k$, $U_{-r+1} = 1$, $U_{-(r-2)} = \cdots = U_0 = 0$ ($r - 1$ zeroes), $U_1 = 1$.

Using the lemma and the pattern made clear by the earlier cases, one could prove the final generalized theorem given below.

Theorem 4.5. Let

$$P_{-(r-2)}(x) = P_{-(r-3)}(x) = \cdots = P_{-1}(x) = P_0(x) = 0, \quad P_1(x) = 1, \quad P_2(x) = x^{r-1},$$

and

$$P_{n+r}(x) = x^{r-1}P_{n+r-1}(x) + x^{r-2}P_{n+r-2}(x) + \cdots + P_n(x) ,$$

and let

$$U_n = P_n(k) .$$

Then, given r non-negative integers $N_1, N_2, \cdots, N_r$, there exists a unique set of integers $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_s, j\}$ such that

$$N_i = \epsilon_1 U_{j+i+1} + \epsilon_2 U_{j+i+2} + \cdots + \epsilon_s U_{j+i+s}$$

for $i = 1, 2, \cdots, r$, where

$$U_i \in \{U_n\}_{-\infty}^{+\infty}$$

and ϵ_i satisfies the constraints $\epsilon_i = 0, 1, 2, \cdots, k^{r-1}$, $-\infty < i < +\infty$; where if $\epsilon_{i+r-2} = k^{r-1}$, then $\epsilon_{i+r-3} = 0, 1, \cdots, k^{r-2}$; if $\epsilon_{i+r-2} = k^{r-1}$ and $\epsilon_{i+r-3} = k^{r-2}, \cdots,$ and $\epsilon_{i+1} = k^2$, $\epsilon_i = k$, then $\epsilon_{i-1} = 0$.

5. CONDITIONS FOR REPRESENTABILITY

In Section 4, a necessary and sufficient condition for representability of an integer pair A, B by Klarner's Theorem was given as

$$\alpha B + A \geq 0, \qquad \alpha = (1 + \sqrt{5})/2 ,$$

where α is the positive root of $\lambda^2 - \lambda - 1 = 0$. Here a proof is provided, as well as statement and proof in the general case.

First, the Fibonacci polynomials have the recursion relation

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$$

and hence the associated polynomial

$$\lambda^2 - x\lambda - 1 = 0$$

with roots λ_1 and λ_2, $\lambda_1 > \lambda_2$, $\lambda_1 = (x + \sqrt{x^2+4})/2$. If $F_1(x)$ is written as a linear combination of the roots, $F_1(x) = A_1\lambda_1 + A_2\lambda_2$, then $F_n(x) = A_1\lambda_1^n + A_2\lambda_2^n$. We consider the limiting ratio of successive Fibonacci polynomials, which becomes

$$\lim_{n\to\infty} \frac{F_{n+1}(x)}{F_n(x)} = \lim_{n\to\infty} \frac{A_1\lambda_1^{n+1} + A_2\lambda_2^{n+1}}{A_1\lambda_1^n + A_2\lambda_2^n} = \lambda_1 = \frac{x + \sqrt{x^2+4}}{2}$$

upon dividing through by λ_1^n, since $|\lambda_2/\lambda_1| < 1$.

Now let $H_0 = A$ and $H_1 = B$, linear combinations of elements of the sequence of Fibonacci polynomials evaluated at $x = k$ as defined in Theorem 4.1, and let $H_n = kH_{n-1} + H_{n-2}$ be the recursion relation for $\{H_n\}$. Then, as the special case of identity (4.6) proved in [2] where $r = 2$,

$$H_{n+1} = H_1 F_{n+1}(k) + H_0 F_n(k)$$

$$\frac{H_{n+1}}{F_n(k)} = \frac{H_1 F_{n+1}(k)}{F_n(k)} + H_0 .$$

For sufficiently large n, we have $H_{n+1}/F_n(k) > 0$. Thus, taking the limit of the expression above as n tends to infinity,

$$0 \le \lambda_1 H_1 + H_0 = B(k + \sqrt{k^2 + 4})/2 + A , \tag{5.1}$$

the condition for representability of an integer pair A, B by Theorem 4.1, with equality only if $A = B = 0$ is vacuously represented. The conditions for Klarner's Theorem follow when $k = 1$.

In the Tribonacci case, we let $H_0 = A$, $H_1 = B$, $H_2 = C$, linear combinations of the elements of the sequence of Tribonacci polynomials evaluated at $x = k$ as defined in Theorem 4.3, and let $H_n = k^2 H_{n-1} + kH_{n-2} + H_{n-3}$ be the recursion relation for the sequence $\{H_n\}$, the same recursion as for the Tribonacci polynomials evaluated at $x = k$, the sequence $\{T_n(k)\}$. Both $\{T_n(k)\}$ and $\{H_n\}$ have, then, the associated polynomial

$$\lambda^3 - k^2\lambda^2 - k\lambda - 1 = 0$$

with roots $\lambda_1, \lambda_2, \lambda_3$, where $\lambda_1 > |\lambda_2| \ge |\lambda_3|$, $k^2 < \lambda_1 < k^3$ for $k \ge 2$, and $1 < \lambda_1 < 2$ for $k = 1$; and λ_1 is the root greatest in absolute value. Analogous to the Fibonacci case, we can prove that

$$\lim_{n\to\infty} \frac{T_{n+m}(k)}{T_n(k)} = \lambda_1^m .$$

Again applying the identity (4.6) from [2], where $r = 3$, we write

$$H_{n+2} = H_2 T_{n+1}(k) + H_1[kT_n(k) + T_{n-1}(k)] + H_0 T_n(k) .$$

Upon division by $T_{n-1}(k)$, for n sufficiently large, $H_{n+2}/T_{n-1}(k) > 0$. Then we evaluate the limit as n approaches infinity to obtain

$$(5.2) \qquad 0 \le \lambda_1^2 H_2 + H_1[k\lambda_1 + 1] + H_0\lambda_1 = \lambda_1^2 C + B[k\lambda_1 + 1] + \lambda_1 A ,$$

with equality only if $A = B = C = 0$, the vacuous representation. Thus, we have the conditions for representability of an integer triple A, B, C in terms of Tribonacci polynomials evaluated at $x = k$ as in Theorem 4.3. The conditions for Theorem 4.2 for representation using Tribonacci numbers follow when $k = 1$.

Now, for representability in the general case, we consider a sequence $\{H_n\}$ having the same recursion as the generalized Fibonacci polynomials $\{P_n(k)\}$,

$$H_{n+r} = k^{r-1}H_{n+r-1} + k^{r-2}H_{n+r-2} + \cdots + H_n ,$$

and take as its initial values $H_0 = N_1$, $H_1 = N_2, \cdots, H_{r-1} = N_r$, the r integers represented in Theorem 4.5. Now, the generalized Fibonacci polynomials evaluated at $x = k$ have the associated polynomial

$$(5.3) \qquad \lambda^r = (k\lambda)^{r-1} + (k\lambda)^{r-2} + \cdots + k\lambda + 1$$

with roots $\lambda_1 > |\lambda_2| \ge |\lambda_3| \ge \cdots \ge |\lambda_r|$, where $k^{r-1} < \lambda_1 < k^{r-1} + 1/k$, $k \ge 2$, and where λ_1 is the root of greatest modulus. (If $k = 1$, then $1 < \lambda_1 < 2$.)

We next prove that there is a root of greatest modulus for (5.3), that the roots are distinct, and that the root of greatest modulus is positive and lies in the interval described.

Lemma 1. Let

$$f(\lambda) = \lambda^r - (k\lambda)^{r-1} - (k\lambda)^{r-2} - \cdots - k\lambda - 1 .$$

Then, for $r \ge 2$ and $k \ge 2$, $f(k^{r-1}) < 0$ and $f(k^{r-1} + 1/k) > 0$.

Proof. Let $\lambda^* = k\lambda$, so that

$$h(\lambda^*) = k^r f(\lambda) = \lambda^{*r} - k^r(\lambda^{*r-1} + \lambda^{*r-2} + \cdots + 1) .$$

Then,

$$\begin{aligned} h(k^r) &= k^{r^2} - k^r(k^{r^2-r} + k^{r^2-2r} + \cdots + 1) \\ &= k^{r^2} - k^{r^2} - k^{r^2-r} - \cdots - k^r < 0 , \end{aligned}$$

and this implies that $f(k^{r-1}) < 0$.

Now, let $\lambda_1 > 1/k$ be a zero of $g(\lambda)$, where

$$g(\lambda) = (\lambda k - 1)f(\lambda) = (\lambda k - 1)\left(\lambda^r - \frac{(k\lambda)^r - 1}{k\lambda - 1}\right)$$

by summing the geometric series formed by all but the first term of $f(\lambda)$. Then,

$$-g(\lambda_1) = \lambda_1^r(k^r + 1 - k\lambda_1) - 1 = 0$$

so that

$$\lambda_1^r(k^r + 1 - k\lambda_1) = 1 .$$

Thus

$$k^r + 1 - k\lambda_1 > 0 ,$$

or

$$\lambda_1 < k^{r-1} + \frac{1}{k} .$$

We note that $k^{r-1} < \lambda_1 < k^{r-1} + 1/k$ for $k \geq 2$, $r > 2$, agrees with $1 < \lambda_1 < 2$ for the case $k = 1$, $r \geq 2$.

Lemma 2. Take $f(\lambda)$ as defined in Lemma 1, and let

$$g(\lambda) = (\lambda k - 1)f(\lambda) = \lambda^{r+1}k - \lambda^r - \lambda^r k^r + 1 .$$

Then, $g(\lambda)$ and $g'(\lambda)$ have no common zeros.

Proof. Since

$$g'(\lambda) = \lambda^{r-1}\,[\lambda k(r + 1) - r(1 + k^r)] ,$$

$\lambda = 0$ is an $(r - 1)$-fold zero of $g'(\lambda)$, and the other root is

$$\lambda = \frac{r}{k(r + 1)}\,(1 + k^r) .$$

We observe that $\lambda = 0$ is not a zero of $g(\lambda)$, and

$$\lambda g'(\lambda) = rg(\lambda) + \lambda^{r+1}k - r .$$

Let λ_0 be a common root of $g'(\lambda) = 0$ and $g(\lambda) = 0$ so that

$$\lambda_0^{r+1}k - r = 0, \qquad \text{or} \qquad \lambda_0^{r+1} = \frac{r}{k} .$$

We note in passing that if $k\lambda_0^{r+1} = r$ and $g(\lambda_0) = 0$, then

$$g(\lambda_0) = \lambda_0^{r+1}k - \lambda_0^r(1 + k^r) + 1 = 0$$

so that

$$g(\lambda_0) = r - \lambda_0^r(1 + k^r) + 1 = 0, \qquad \text{or} \qquad \lambda_0^r = \frac{r + 1}{1 + k^r} .$$

We now solve for λ_0 :

$$\lambda_0^{r+1} = \frac{(\lambda_0)(r+1)}{1+k^r} = \frac{r}{k}$$

or

$$\text{(a)} \qquad \lambda_0 = \frac{r(1+k^r)}{(r+1)k} .$$

We now show that $k\lambda_0^{r+1} = r$ is inconsistent with (a), by demonstrating that

$$\left(\frac{r(k^r+1)}{(r+1)k}\right)^{r+1} > \frac{r}{k} .$$

For $k \geq 2$, $r \geq 3$, $k^{r^2-1}/r > 4$ and

$$1 < \left(\frac{1+k^r}{k^r}\right)^{k^r+1} < 4$$

so that

$$1 < \left(\frac{1+k^r}{k^r}\right)^{r+1}$$

$$4 < \left(\frac{k^{r^2-1}}{r}\right) \cdot \left(\frac{1+k^r}{k^r}\right)^{r+1}$$

while

$$4 > \left(\frac{r+1}{r}\right)^{r+1} > e .$$

The fact that

$$\left(\frac{k^{r^2-1}}{r}\right) \cdot \left(\frac{1+k^r}{k^r}\right)^{r+1} > \left(\frac{r+1}{r}\right)^{r+1}$$

is equivalent to the stated inequality. Thus we conclude that there are no common zeros between the functions $g(\lambda)$ and $g'(\lambda)$, for if there would be at least one repeated root λ_0, then the two expressions for λ_0^{r+1} would be equal, which has been shown to be impossible.

Comments. For all integers $r \geq 2$ and $k \geq 1$, the

Theorem. The roots of

$$\lambda^r = \lambda^{r-1} + \lambda^{r-2} + \cdots + \lambda + 1$$

are distinct. The root λ_1, of greatest modulus, lies in the interval $1 < \lambda_1 < 2$, and the remaining $r - 1$ roots $\lambda_2, \lambda_3, \cdots, \lambda_r$ satisfy $|\lambda_j| < 1$ for $j = 2, 3, \cdots, r$.

was proved by E. P. Miles, Jr., in [4]. The case $k \geq 2$ and $r = 2$ is very easy to prove, involving only a quadratic. The general case for $k \geq 2$ and $r \geq 3$ now follows.

Theorem 5.1. For $r \geq 3$, $k \geq 2$, the roots $\lambda_1, \lambda_2, \cdots, \lambda_r$ of the polynomial

$$f(\lambda) = \lambda^r - (k\lambda)^{r-1} - (k\lambda)^{r-2} - \cdots - k\lambda - 1$$

are distinct, and λ_1, the root of greatest modulus, satisfies

$$k^{r-1} < \lambda_1 < k^{r-1} + \frac{1}{k} .$$

Proof. Let

$$g(\lambda) = (\lambda k - 1)f(\lambda) = \lambda^{r+1}k - \lambda^r - \lambda^r k^r + 1 .$$

Clearly, $g(\lambda)$ has the same zeros as $f(\lambda)$ except that $g(\lambda) = 0$ also when $\lambda = 1/k$. By Lemma 2, the polynomial $g(\lambda)$ has no repeated zeros and thus for $k \geq 2$, $r \geq 3$, the polynomial $f(\lambda)$ has no repeated zeros.

We now show that the root λ_1 of Lemma 1,

$$k^{r-1} < \lambda_1 < k^{r-1} + \frac{1}{k} ,$$

is the zero of greatest modulus for the polynomial $f(\lambda)$. We make use of the theorem appearing in Marden [5]:

Theorem (32, 1) (Montel): At least p zeros of the polynomial

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

lie in the circular disk

$$|z| < 1 + \max\left|\frac{a_j}{a_n}\right|^{\frac{1}{n-p+1}} , \qquad j = 0, 1, 2, \cdots, p .$$

As applied to our $f(\lambda)$,

$$f(\lambda) = \lambda^r - (k\lambda)^{r-1} - (k\lambda)^{r-2} - \cdots - k\lambda - 1 ,$$

$a_r = 1$ and $a_j = k^j$, $j = 0, 1, \cdots, r - 1$. Thus $(r - 1)$ of the zeros of $f(\lambda)$ lie inside the disk

$$|\lambda| < 1 + k^{(r-1)/2} .$$

To show that $\lambda_1 > |\lambda|$, we simply compare the two. From Lemma 1,

$$k^{r-1} < \lambda_1 < k^{r-1} + 1/k ,$$

$$|\lambda| < 1 + k^{(r-1)/2} .$$

The quadratic $x^2 - x - 1 > 0$ if $x > (1 + \sqrt{5})/2$; thus, $k^{(r-1)/2} = x > (1 + \sqrt{5})/2$ if $k \geq 2$ and $r \geq 3$. Therefore, the $(r - 1)$ zeros of $f(\lambda)$ distinct from λ_1 have modulus less than that of λ_1. We conclude that $f(\lambda)$ does indeed have a positive root and this root is the one of greatest modulus.

Corollary 5.1.1. For all real numbers $k \geq 2$ and all positive integers $r \geq 2$, the polynomial $f(\lambda)$ of Theorem 5.1 has distinct zeroes and λ_1, the zero of greatest modulus, is positive and satisfies $k^{r-1} < \lambda_1 < k^{r-1} + 1/k$.

Corollary 5.1.2. The only positive root λ_1 of the polynomial $f(\lambda)$ of Theorem 5.1 lies in the interval

$$k^{r-1} + \frac{1}{k} - \frac{1}{k^{r^2-r+1}} < \lambda_1 < k^{r-1} + \frac{1}{k} .$$

Proof. We have only to show that

$$\lambda_1 > k^{r-1} + \frac{1}{k} - \frac{1}{k^{r^2-r+1}} = a .$$

Calculating $f(a)$ from the following form,

$$f(\lambda) = \lambda^r - \frac{(k\lambda)^r - 1}{k\lambda - 1} = \frac{k\lambda^{r+1} - (k^r + 1)\lambda^r + 1}{k\lambda - 1}$$

it is not difficult to show that $f(a) < 0$. But $f(\lambda) > 0$ whenever $\lambda > \lambda_1$, and λ_1 is the only positive root. Also, it is not difficult to show that $a > 0$. Therefore, we must have $\lambda_1 > a$.

Corollary 5.1.2 still yields $1 < \lambda_1 < 2$ for $k = 1$, $r \geq 2$. For $k = 10$ and $r = 10$, the root can vary only in an interval $\Delta = 1/10^{91}$; if $k = 10$ and $r = 100$, then $\Delta = 1/10^{9901}$ making an extremely accurate approximation for large values of r.

The following improved proof of Theorem 5.1 was given by A. P. Hillman [7]. First,

$$f(\lambda) = \lambda^r - \frac{(k\lambda)^r - 1}{k\lambda - 1} = \frac{k\lambda^{r+1} - (k^r + 1)\lambda^r + 1}{k\lambda - 1} = \frac{k\lambda^r(\lambda - k^{r-1}) - (\lambda^r - 1)}{k\lambda - 1}$$

$$f(k^{r-1}) = -\frac{k^{r(r-1)} - 1}{k^r - 1} < 0$$

and

$$f(k^{r-1} + (1/k)) = \frac{k(k^{r-1} + (1/k))^r \cdot (1/k) - [(k^{r-1} + (1/k))^r - 1]}{k^r} = \frac{1}{k^r} > 0.$$

It now follows from the Intermediate Value Theorem that $f(\lambda_1) = 0$ for some λ_1 with $k^{r-1} < \lambda_1 < k^{r-1} + 1/k$. But Descartes' Rule of Signs tells us that $f(\lambda) = 0$ has only one positive root. Hence, λ_1 is the only positive root. Since the coefficient of the highest power, λ^r, is positive in $f(\lambda)$, we know that $f(\lambda) > 0$ for λ very large. But $f(\lambda)$ does not change sign for $\lambda > \lambda_1$. Hence $f(\lambda) > 0$ for $\lambda > \lambda_1$.

Now let $|\lambda| = p$ with $p > \lambda_1$. Then $p^r - (kp)^{r-1} - (kp)^{r-2} - \cdots - kp - 1 > 0$, or $p^r > (kp)^{r-1} + (kp)^{r-2} + \cdots + kp + 1$, and

$$\begin{aligned} |\lambda^r| = p^r > (kp)^{r-1} + \cdots + kp + 1 &= |(k\lambda)^{r-1}| + \cdots + |k\lambda| + 1 \\ &\geq |(k\lambda)^{r-1} + \cdots + k\lambda + 1| . \end{aligned}$$

Hence, $\lambda^r \neq (k\lambda)^{r-1} + (k\lambda)^{r-2} + \cdots + k\lambda + 1$ and so $f(\lambda) \neq 0$ for $\lambda > \lambda_1$.

Next let $|\lambda| = \lambda_1$ with $\lambda \neq \lambda_1$. Since

$$|z_1| + |z_2| + \cdots + |z_n| > |z_1 + z_2 + \cdots + z_n|$$

if the z_i are not all on the same ray from the origin,

$$\begin{aligned} |\lambda^r| = \lambda_1^r &= (k\lambda_1)^{r-1} + \cdots + k\lambda_1 + 1 \\ &= |(k\lambda)^{r-1}| + \cdots + |k\lambda| + 1 > |(k\lambda)^{r-1} + \cdots + k\lambda + 1| . \end{aligned}$$

Thus, for $|\lambda| = \lambda_1$, $\lambda \neq \lambda_1$, we have $\lambda^r \neq (k\lambda)^{r-1} + \cdots + k\lambda + 1$, or $f(\lambda) \neq 0$.

All that remains is to show that λ_1 is not a multiple root of $f(\lambda) = 0$, i.e., not a root of $f'(\lambda) = 0$. Since $f(\lambda_1) = 0$, we have

$$(k^r + 1)\lambda_1^r = k\lambda_1^{r+1} + 1 .$$

Then

$$\begin{aligned} f'(\lambda_1) = \frac{(r+1)k\lambda_1^r - r(k^r + 1)\lambda_1^{r-1}}{k\lambda_1 - 1} &= \frac{(r+1)k\lambda_1^{r+1} - r(k^r + 1)\lambda_1^r}{\lambda_1(k\lambda_1 - 1)} \\ &= \frac{(2rk + k + r)\lambda_1^{r+1}}{\lambda_1(k\lambda_1 - 1)} . \end{aligned}$$

Hence, $f'(\lambda_1) \neq 0$ and the proof is finished.

Corollary. Theorem 5.1 holds for any $k > 0$.

Proof. Examine the Hillman proof of the theorem and see the fact that $k \geq 2$ was not explicitly used as in the earlier proof. This extends Theorem 5.1 to include E. P. Miles' theorem.

Theorem 5.1 states that the zeros of $f(\lambda)$ are distinct. Something of this kind is needed since if a root, say λ_2, is repeated $(r - 1)$ times, then

$$Q_n = A_1\lambda_1^n + \lambda_2^n(B_2 n^{r-2} + B_3 n^{r-3} + \cdots + B_r)$$

and the existence of

$$\lim_{n \to \infty} \frac{\lambda_2^{n+1}(B_2 n^{r-2} + B_3 n^{r-3} + \cdots + B_r)}{\lambda_1^n}$$

may be in doubt or at least it raises some questions.

Now, for the generalized Fibonacci polynomials $\{P_n(k)\}$, since we can write

$$P_n(k) = A_1\lambda_1^n + A_2\lambda_2^n + \cdots + A_r\lambda_r^n ,$$

a linear combination of the roots of the associated polynomial (5.3),

$$\frac{P_{n+m}(k)}{P_n(k)} = \frac{A_1\lambda_1^{n+m} + A_2\lambda_2^{n+m} + \cdots + A_r\lambda_r^{n+m}}{A_1\lambda_1^n + A_2\lambda_2^n + \cdots + A_r\lambda_r^n} .$$

Upon division by λ_1^n, since $|\lambda_i/\lambda_1| < 1$, $i = 2, 3, \cdots, r$,

$$\lim_{n \to \infty} \frac{P_{n+m}(k)}{P_n(k)} = \lambda_1^m$$

so that the ratio of a pair of successive generalized Fibonacci polynomials evaluated as $x = k$ approaches the greatest positive root of its associated polynomial as n approaches infinity.

Now, the following was proved as identity (4.6) in [2]:

$$\begin{aligned}(5.4)\qquad H_{n+r-1} = {} & H_{r-1}P_{n+1}(k) + H_{r-2}[k^{r-2}P_n(k) + k^{r-3}P_{n-1}(k) \\ & \qquad + \cdots + P_{n-r+2}(k)] \\ & + H_{r-3}[k^{r-3}P_n(k) + k^{r-4}P_{n-1}(k) \\ & \qquad + \cdots + P_{n-r+3}(k)] \\ & + \cdots \\ & + H_1[xP_n(k) + P_{n-1}(k)] + H_0P_n(k) .\end{aligned}$$

Upon division by $P_{n-r+2}(k)$, if $H_{n+r-1}/P_{n-r+2}(k) > 0$ for all sufficiently large values of n, (5.4) becomes

$$\begin{aligned} 0 \leq H_{r-1}\lambda_1^{r-1} &+ H_{r-2}[(k\lambda_1)^{r-2} + (k\lambda_1)^{r-3} + \cdots + k\lambda_1 + 1] \\ &+ \lambda_1 H_{r-3}[(k\lambda_1)^{r-3} + (k\lambda_1)^{r-4} + \cdots + k\lambda_1 + 1] \\ &+ \lambda_1^2 H_{r-4}[(k\lambda_1)^{r-4} + (k\lambda_1)^{r-5} + \cdots + k\lambda_1 + 1] \\ &+ \cdots + \lambda_1^{r-3}H_1(k\lambda_1 + 1) + \lambda_1^{r-2}H_0 \;. \end{aligned}$$

Thus, the representability condition for Theorem 4.5, for the generalized Fibonacci polynomials evaluated at $x = k$, becomes

$$\begin{aligned} 0 \leq N_r\lambda_1^{r-1} &+ N_{r-1}[(k\lambda_1)^{r-2} + (k\lambda_1)^{r-3} + \cdots + k\lambda_1 + 1] \\ &+ \lambda_1 N_{r-2}[(k\lambda_1)^{r-3} + (k\lambda_1)^{r-4} + \cdots + k\lambda_1 + 1] \\ &+ \lambda_1^2 N_{r-3}[(k\lambda_1)^{r-4} + (k\lambda_1)^{r-5} + \cdots + k\lambda_1 + 1] \\ &+ \cdots + \lambda_1^{r-3}N_2(k\lambda_1 + 1) + \lambda_1^{r-2}N_1 \;, \end{aligned}$$

where λ_1 is the positive root of greatest absolute value of the associated polynomial (5.3), with equality only if $N_1 = N_2 = \cdots = N_r = 0$, the vacuous representation.

When $k = 1$, we have the representation conditions for r integers $N_1, N_2, \cdots, N_r$ in terms of the r-bonacci numbers as in Theorem 4.4.

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