ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\quad F_0 = 0,\quad F_1 = 1 \quad\text{and}\quad L_{n+2} = L_{n+1} + L_n,\quad L_0 = 2,\quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-274 Proposed by C. B. A. Peck, State College, Pennsylvania.

Approximate $(\sqrt{5} - 1)/2$ to within 0.002 using at most three distinct familiar symbols. (Each symbol may represent a number or an operation and may be repeated in the expression.)

B-275 Proposed by Warren Cheves, Littleton, North Carolina.

Show that

$$F_{mn} = L_m F_{m(n-1)} + (-1)^{m+1} F_{m(n-2)}\ .$$

B-276 Proposed by Graham Lord, Temple University, Philadelphia, Pennsylvania.

Find all the triples of positive integers m, n, and x such that

$$F_h = x^m \quad\text{where}\quad h = 2^n \quad\text{and}\quad m > 1\ .$$

B-277 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Ill.

Prove that $L_{2n(2k+1)} \equiv L_{2n} \pmod{F_{2n}}$.

B-278 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Ill.

Prove that $L_{(2n+1)(4k+1)} \equiv L_{2n+1} \pmod{F_{2n+1}}$.

B-279 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Find a closed form for the coefficient of x^n in the Maclaurin series expansion of

$$(x + 2x)/(1 - x - x^2)^2\ .$$

SOLUTIONS

SEVEN DO'S FOR TWO SUSY'S

B-250 Proposed by Guy A.R. Guillotte, Montreal, Quebec, Canada.

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  DO
 YOU
LIKE
----
SUSY
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In this alphametic, each letter stands for a particular but different digit, nine digits being shown here. What do you make of the perfect square sum SUSY ?

Solution by Raymond E. Whitney, Lock Haven State College, Lock Haven, Pa.

There are four possible SUSY's. They are 2025, 3136, 6561, 8281. SUSY = 2025 leads to the six solutions shown below:

```
  76     86     49     89     98     48
 560    560    590    590    580    580
1389   1379   1386   1346   1347   1397
----   ----   ----   ----   ----   ----
2025   2025   2025   2025   2025   2025
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SUSY = 3136 leads to one solution:

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  57
 671
2408
----
3136 .
```

The other two 4-digit numbers lead to no solutions. Thus the likelihood is that SUSY = 2025 and she is definitely 2025 or 3136.

Also solved by Richard Blazej, Donald Braffitt, Paul S. Bruckman, Juliana D. Chan, Warren Cheves, Herta T. Freitag, Ralph Garfield, Myron Hlynka, J. A. H. Hunter, John W. Milsom, C. B. A. Peck, Jim Pope, Richard W. Sielaff, Charles W. Trigg, Lawrence Williams, David Zeitlin, and the Proposer.

FAIR GAME

B-251 Proposed by Paul S. Bruckman, San Rafael, California.

A and B play a match consisting of a sequence of games in which there are no ties. The odds in favor of A winning any one game is m. The match is won by A if the number of games won by A minus the number won by B equals 2n before it equals -n. Find m in terms of n given that the match is a fair one, i. e., the probability is 1/2 that A will win the match.

Solution by Dennis Staples, The American School in Japan, Tokyo, Japan.

The game specifies that A wins whenever A's wins - B's wins reaches 2n before it reaches -n. Said another way, A wins whenever A's wins - B's wins + n reaches 3n before it reaches 0.

Recalling the notion of Markov chains, let u(i) be the probability that A reaches 3n (in other words, wins), given that i = A's wins - B's wins + n. Using techniques designed for solution of Markov chain problems, it can be found that

$$u(i) = \frac{\left(\frac{1-m}{m}\right)^i - \left(\frac{1-m}{m}\right)^{3n}}{1 - \left(\frac{1-m}{m}\right)^{3n}} .$$

Since A and B begin competition when i = n, and since A's chances of winning are to be 1/2,

$$\frac{1}{2} = \frac{\left(\frac{1-m}{m}\right)^n - \left(\frac{1-m}{m}\right)^{3n}}{1 - \left(\frac{1-m}{m}\right)^{3n}}$$

$$1 - \left(\frac{1-m}{m}\right)^{3n} = 2\left(\frac{1-m}{m}\right)^n - 2\left(\frac{1-m}{m}\right)^{3n}$$

$$\left(\frac{1-m}{m}\right)^{3n} - 2\left(\frac{1-m}{m}\right)^n + 1 = 0 .$$

This is a rather familiar equation, and it can be easily shown that the roots, (1 - m)/m, are equal to the following when n = 1 :

$$-1, \quad \frac{-1+\sqrt{5}}{2}, \quad \text{or} \quad \frac{-1-\sqrt{5}}{2} .$$

Of these values, only $(-1 + \sqrt{5})/2 = 0.618\cdots$ is acceptable in the case we are considering. Thus, when n = 1,

$$\frac{1-m}{m} = 0.618 \cdots$$

$$1 - m = (0.618 \cdots) m$$

$$m = \frac{1}{1.618\cdots} = 0.618 \cdots .$$

For the general case,

$$m = \left(\frac{1}{1.618\cdots}\right)^{1/n} = (0.618\cdots)^{1/n} .$$

Also solved by Ralph Garfield and the Proposer.

SOMEWHAT ALTERNATING SUM OF TRINOMIALS

B-252 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Prove that

$$\sum_{i+j+k=n} \frac{(-1)^k}{i!\,j!\,k!} = \frac{1}{n!} .$$

Solution by Harvey J. Hindin, Dix Hills, New York.

The multinomial theorem (G. Chrystal, Textbook of Algebra, Part 2, Dover Reprint, New York, 1961, page 12), may be stated as:

(1) $$(x + y + z)^n = \sum_{i+j+k=n} \frac{n!}{i!\,j!\,k!}\, x^i y^j z^k .$$

If we let $x = y = 1$, and $z = -1$, we have:

(2) $$(1 + 1 - 1)^n = 1 = n! \sum_{i+j+k=n} \frac{(-1)^k}{i!\,j!\,k!}$$

or

(3) $$\sum_{i+j+k=n} \frac{(-1)^k}{i!\,j!\,k!} = \frac{1}{n!} \qquad \text{Q.E.D.}$$

Problem 34, page 20 of Chrystal is similar.

Also solved by Paul S. Bruckman, Michael Capobianco, Timothy B. Carroll, Herta T. Freitag, Ralph Garfield, Lawrence D. Gould, Myron Hlynka, Graham Lord, C. B. A. Peck, Raymond E. Whitney, David Zeitlin, and the Proposer.

TRINOMIAL EXPANSION WITH F'S AND L'S

B-253 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Prove that

$$\sum_{i+j+k=n} \frac{(-1)^k L_{j+2k}}{i!\,j!\,k!} = 0 = \sum_{i+j+k=n} \frac{(-1)^k F_{j+2k}}{i!\,j!\,k!} .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

In the trinomial expansion

$$\sum_{i+j+k=n} x^i y^j z^k \binom{n}{i,j,k} = (x + y + z)^n ,$$

where

$$\binom{n}{i,j,k} = n!/i!\,j!\,k!$$

with $i + j + k = n$, let $x = 1$, $y = \alpha(\beta)$, $z = -\alpha^2(-\beta^2)$. From the Binet formulas, the two expressions are proportional to $(1 + \alpha - \alpha^2)^n \pm (1 + \beta - \beta^2)^n = 0^n \pm 0^n = 0$.

Comment. A number of solvers pointed out that the F's or L's could be replaced by generalized Fibonacci numbers.

Also solved by Paul S. Bruckman, Timothy B. Carroll, Herta T. Freitag, Ralph Garfield, Harvey J. Hindin, Graham Lord, David Zeitlin, and the Proposer.

MORE OR LESS LUCAS

B-254 Proposed by Clyde A. Bridger, Springfield, Illinois.

Let $A_n = a^n + b^n + c^n$ and $B_n = d^n + e^n + i^n$, where a, b, and c are the roots of $x^3 - 2x - 1$ and d, e, and f are the roots of $x^3 - 2x^2 + 1$. Find recursion formulas for the A_n and for the B_n. Also express B_n in terms of A_n.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

The roots of $x^3 - 2x - 1$ are $-1, \alpha, \beta$ and of $x^3 - 2x^2 + 1$ are $1, \alpha, \beta$ where α, β have their usual (Fibonacci) meaning. Hence if L_n is the n^{th} Lucas number, then

$$A_n = (-1)^n + L_n \quad \text{and} \quad B_n = 1 + L_n .$$

Consequently from the properties of L_n for $n \ge 0$:

$$A_{n+3} = 2A_{n+1} + A_n ,$$
$$B_{n+3} = 2B_{n+2} - B_n ,$$

and

$$B_n = \begin{cases} A_n & n \text{ even} \\ A_n + 2 & n \text{ odd} \end{cases}$$

or

$$B_n = A_n + 1 - (-1)^n \qquad \text{Q.E.D.}$$

Also solved by Richard Blazej, Paul S. Bruckman, Timothy B. Carroll, Herta T. Freitag, Ralph Garfield, Robert McGee and Juliana D. Chan, Raymond E. Whitney, Gregory Wulczyn, David Zeitlin, and the Proposer.

FIBONACCI CONVOLUTION REVISITED

B-255 Proposed by L. Carlitz and Richard Scoville, Duke University, Durham, North Carolina.

Show that

$$\sum_{2k \le n} k\binom{n-k}{k} = \sum_{k=0}^{n} F_k F_{n-k} = [(n-1)F_{n+1} + (n+1)F_{n-1}]/5 .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

The l.h. result is proved by Carlitz in the Fibonacci Quarterly, Vol. VII, No. 3, pp. 285-286 (proposed by Hoggatt as H-131 in Vol. VI, No. 2, p. 142), so we confine ourselves to the r.h. result (stated by Wall in Vol. I, No. 4, p. 28). The result for $n = 0$ is just $F_0F_{0-0} = 0\cdot 0 = 0 = [-1\cdot 1 + 1\cdot 1]/5 = [(0-1)F_{0+1} + (0+1)F_{0-1}]/5$. and for $n = 1$ is $F_0F_{1-0} + F_1F_{1-1} = 0\cdot 1 + 1\cdot 0 = 0 = [0\cdot 2 + 2\cdot 0]/5 = [(1-1)F_{1+1} + (1+1)F_{1-1}]/5$. Suppose the result true for all n up to some $m \ge 1$. Then

$$\sum_{k=0}^{m+1} F_k F_{m+1-k} = \sum_{k=0}^{m} F_k F_{m-k} + \sum_{k=0}^{m-1} F_k F_{m-1-k} + F_{m+1}F_{m+1-(m+1)} + F_m F_{m-1-m}$$

$$= [(m-1)F_{m+1} + (m+1)F_{m-1} + (m-2)F_m + mF_{m-2} + F_m 5]/5$$

$$= [mF_{m+2} + (m+2)F_m]/5 ,$$

so that by the Second Principle of Finite Induction, the right-hand result is true.

Also solved by Paul S. Bruckman, Timothy B. Carroll, Herta T. Freitag, Ralph Garfield, Phil Tracy, and the Proposer.