THE GOLDEN ELLIPSE

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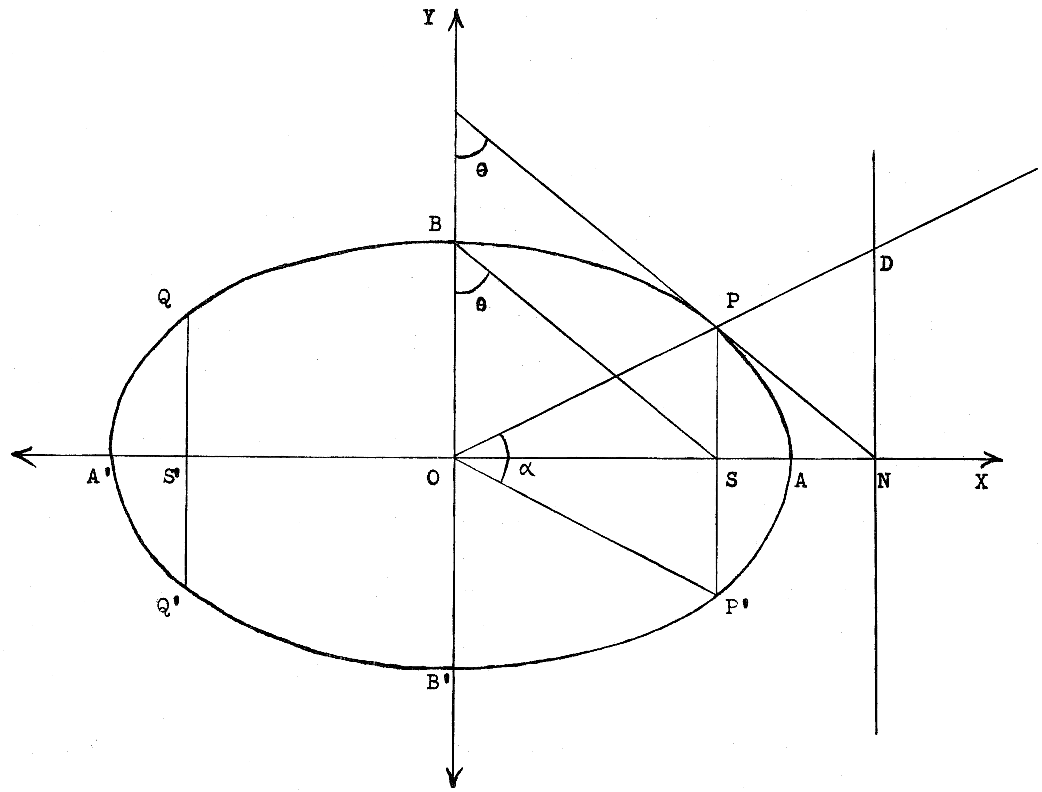
The ellipse in which the ratio of the major to the minor axis (a/b) is the golden ratio φ — the "divine proportion" of Renaissance mathematicians [1] — has interesting properties. In the figure, let OA = a = φ units, OB = b = 1 unit. Clearly, if a rectangle be circumscribed about the ellipse, having its sides parallel to the axes, it will be what has been called the golden rectangle, frequently realized in ancient Greek architecture.



The Golden Ellipse
$a/b = \varphi$

The modern symbol for "golden section," as it was called in the nineteenth century, is the Greek letter phi: φ, and φ and φ' are the solutions of the equation $x^2 - x - 1 = 0$.

$$\varphi = (1 + \sqrt{5})/2 = -1.6180 \cdots,$$

$$\varphi' = (1 - \sqrt{5})/2 = -0.6180\cdots,$$

so that $\varphi + \varphi' = 1$ and $\varphi\varphi' = -1$.

Now, the eccentricity e of an ellipse is given by

$$b^2 = a^2(1 - e^2)$$

so that the eccentricity of the golden ellipse is

(1) $$e = 1/\sqrt{\varphi} = \sqrt{-\varphi'} \quad .$$

Using the familiar notation of the figure (e.g., S and S' are the foci), from the known properties of the ellipse, we may write the following equations:

(2) $$OS = ae = \sqrt{\varphi}$$

(3) $$BS = \sqrt{(b^2 + a^2e^2)} = a = \varphi \quad .$$

If $\angle OBS = \theta$,

(4) $$\sec\theta = a/b = \varphi \quad .$$

If ON is perpendicular to the directrix ND,

(5) $$ON = a/e = \varphi^{3/2}$$

(6) $$SN = ON - OS = a/e - ae = 1/\sqrt{\varphi} \quad .$$

A property of any ellipse may be stated thus: the minor axis is the geometric mean of the major axis and the latus rectum; that is, if L is the length of the semi-latus rectum, $aL - b^2$. Hence, for the golden ellipse,

(7) $$L = b^2/a = 1/\varphi = -\varphi' \quad .$$

Thus

$$a : b : c = \varphi : 1 : -\varphi' = \varphi^2 : \varphi : 1 \quad .$$

From Eqs. (2), (5), and (6),

$$ON/OS = 1/e^2 = \varphi \qquad \text{and} \qquad OS/SN = ae/(a/e - ae) = 1/\varphi = -\varphi' \quad .$$

Hence, the focus S divides ON in the Golden Ratio.

Again, if PP' is the latus rectum,

(8) $$OP^2 = OS^2 + SP^2 = a^2e^2 + b^4/a^4 = \varphi + \varphi'^2 = 2 \quad .$$

Using the cosine rule for $\Delta POP'$, it may be deduced from this that the latus rectum subtends at the center O an angle α given by $\cos\alpha = 1/\varphi$, so that, from (4),

$$\alpha = \theta . \tag{9}$$

Another property of the ellipse has it that a tangent at P passes through N and that $\cot \angle SPN = e$. Since $\cot\theta = b/ae$, it follows in the case of the golden ellipse that

$$\cot \angle SPN = 1/\sqrt{\varphi} \qquad \text{and} \qquad \cot\theta = 1/\sqrt{\varphi} ,$$

so that $\angle SPN = \theta$. Thus, MPSB is a parallelogram, and

$$MP = BS = a = \varphi . \tag{10}$$

Moreover, since MP/PN and OS/SN = φ,

$$PN = 1 \qquad \text{and} \qquad MN = \varphi + 1 = \varphi^2 , \tag{11}$$

and P divides MN in the Golden Ratio.

It is easily shown that $OM = \varphi$ so that M lies on the auxiliary circle of the ellipse and ΔPOM is isosceles. Moreover, if OP produced intersects the directrix in D, $ND = b^2/ae^2 = 1$, BDNO is a rectangle, and P divides OD in the Golden Ratio.

The interested reader may, by searching, discover for himself many other hiding places where the Golden Ratio is lurking in this ellipse. For example, superimpose on this ellipse a second, similar ellipse, with center O but rotated through a right angle. Draw a common tangent cutting OY, OX in R, S respectively, to touch the ellipse in T_1 and T_2. Examine the ratios of the several segments of RS.

Reference

1. H. E. Huntley, "The Golden Cuboid," Fibonacci Quarterly, Vol. 2, No. 3 (Oct. 1964), p. 184.