# ADVANCED PROBLEMS AND SOLUTIONS 

## Edited by

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.
H-237 Proposed by D. A. Millin, High School Student, Annville, Pennsy/vania.
Prove

$$
\sum_{k=0}^{\infty} \frac{1}{F_{2^{k}}}=\frac{7-\sqrt{5}}{2} .
$$

H-238 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Sum the series

$$
S \equiv \sum_{m, n, p=0}^{\infty} x^{m} y^{n} z^{p}
$$

where the summation is restricted to $m_{s} n_{p} p$ such that

$$
m \leqslant n+p, \quad n \leqslant p+m, \quad p \leqslant m+n
$$

## SOLUTIONS

## FIBONACCI COMBINATION

H-202 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Put

$$
\left\{\begin{array}{c}
k \\
j
\end{array}\right\}=\frac{F_{k} F_{k-1} \cdots F_{k-j+1}}{F_{1} F_{2} \cdots F_{j}},\left\{\begin{array}{c}
k \\
0
\end{array}\right\}=1
$$

Show that
(*)

## Solution by the Proposer.

1. We use the well known identity
(1)

$$
\sum_{j=0}^{k}(-1)^{j}\left[\begin{array}{l}
k \\
j
\end{array}\right] q^{1 / 2(j-1)} x^{j}=\prod_{j=0}^{k-1}\left(1-q^{j} x\right)
$$

where

$$
\left[\begin{array}{l}
k \\
j
\end{array}\right]=\frac{\left(1-q^{k}\right)\left(1-q^{k-1}\right) \cdots\left(1-q^{k-j+1)}\right.}{(1-q)\left(1-q^{2}\right) \cdots\left(1-q^{i}\right)}
$$

Put $q=a / \beta$, where $a+\beta=1, a \beta=-1$. It is easily verified that

$$
\left[\begin{array}{l}
k \\
j
\end{array}\right] \rightarrow \frac{\left(\beta^{k}-a^{k}\right)\left(\beta^{k-1}-a^{k-1}\right) \ldots\left(\beta^{k-j+1}-a^{k-j+1}\right)}{(\beta-a)\left(\beta^{2}-a^{2}\right) \ldots\left(\beta^{j}-a^{j}\right)} \beta^{j^{2}-j k}=\left\{\begin{array}{l}
k \\
j
\end{array}\right\} \beta^{j^{2}-j k} .
$$

Next, replace $k$ by $2 k$ and $x$ by $a^{1-k} \beta^{k} x$. Then (1) becomes

$$
\prod_{j=0}^{2 k}\left(\beta^{j}-a^{j-k+1} \beta^{k-j} x\right)=\sum_{j=0}^{2 k}(-1)^{j}\left\{\begin{array}{c}
2 k  \tag{2}\\
j
\end{array}\right\}(a \beta)^{1 / 2(j+1)-j k} x_{x}^{j}
$$

Since

$$
\begin{aligned}
\prod_{j=1}^{k}\left(a^{j-1}-\beta^{j} x\right)\left(\beta^{j-1}-a^{j} x\right) & =(a \beta)^{1 / 2 k(k-1)} \prod_{j=1}^{k}\left(1-a^{-j+k} \beta^{j} x\right)\left(1-a^{j} \beta^{-j+1} x\right) \\
& =(a \beta)^{1 / 2 k(k-1)} \prod_{j=0}^{k-1}\left(1-a^{j} \beta^{k-j} x\right)\left(1-a^{k-j} \beta^{j-k+1} x\right) \\
& =(a \beta)^{1 / 2(k-1)} \prod_{j=0}^{2 k-1}\left(1-a^{j-k+1} \beta^{k-j} x\right)
\end{aligned}
$$

(2) reduces to

$$
\begin{aligned}
\sum_{j=0}^{2 k}(-1)^{1 / 2 j(j-1)+j k}\left\{\begin{array}{c}
2 k \\
j
\end{array}\right\} x^{j} & =(-1)^{1 / 2 k(k-1)} \prod_{i=1}^{k}\left(a^{j-1}-\beta^{j} x\right)\left(\beta^{j-1}-a^{j} x\right) \\
& =(-1)^{1 / 2 k(k-1)} \prod_{j=1}^{k}\left((-1)^{j-1}-L_{2 j-1} x+(-1)^{j} x^{2}\right)
\end{aligned}
$$

Hence for $x=1$ we get

$$
\sum_{j=0}^{2 k}(--1)^{1 / 2 j(j-1)+j k}\left\{\begin{array}{c}
2 k  \tag{3}\\
j
\end{array}\right\}=(-1)^{1 / 2 k(k+1)} \prod_{j=1}^{k} L_{2 j-1}
$$

while for $x=-1$,
(4)

$$
\sum_{j=0}^{2 k}(-1)^{1 / 2 j(j+1)+j k}\left\{\begin{array}{c}
2 k \\
j
\end{array}\right\}=(-1)^{1 / 2 k(k+1)} \prod_{j=1}^{k} L_{2 j-1}
$$

Finally, replacing $j$ by $j+k$, (3) and (4) become

$$
\sum_{j=-k}^{k}(-1)^{1 / 2(j+k)(j+k-1)}\left\{\begin{array}{c}
2 k \\
j+k
\end{array}\right\}=(-1)^{k} \prod_{j=1}^{k} L_{2 j-1}
$$

$$
\sum_{j=-k}^{k}(-1)^{1 / 2(j+k)(j+k+1)}\left\{\begin{array}{c}
2 k \\
j+k
\end{array}\right\}=\prod_{j=1}^{k} L_{2 j-1}
$$

respectively. This completes the proof of (*).
2. To prove (**), we use Gauss's identity
(5)

$$
\sum_{j=0}^{2 k}(-1)^{j}\left[\begin{array}{c}
2 k \\
j
\end{array}\right]=\prod_{j=1}^{k}\left(1-q^{2 j-1}\right)
$$

(for proof see for example G.B. Mathews, Theory of Numbers, Stechert, New York, 1927, p. 209). Replacing q by $a / \beta$; we find that (5) reduces to

$$
\sum_{j=0}^{2 k}(-1)^{j}\left\{\begin{array}{c}
2 k  \tag{6}\\
j
\end{array}\right\} \beta^{(j-k)^{2}}=(-1)^{k}(a-\beta)^{k} \prod_{j=1}^{k} F_{2 j-1}
$$

Similarly, if $q$ is replaced by $\beta / a$, we get

$$
\sum_{j=0}^{2 k}(-1)^{j}\left\{\begin{array}{c}
2 k  \tag{7}\\
i
\end{array}\right\} a^{(j-k)^{2}}=(-1)^{k}(\beta-a)^{k} \prod_{j=1}^{k} F_{2 j-1}
$$

When $k$ is even, we add (6) to (7) to get

$$
\sum_{j=0}^{2 k}(-1)^{j}\left\{\begin{array}{c}
2 k \\
j
\end{array}\right\} L_{(j-k) 2}=2.5^{1 / 2 k} \prod_{j=1}^{k} F_{2 j-1}
$$

When $k$ is odd, we subtract (6) from (7) and get

$$
\sum_{j=0}^{2 k}(-1)^{j}\left\{\begin{array}{c}
2 k \\
j
\end{array}\right\} F_{(j-k)^{2}}=2 \cdot 5^{1 / 2(k-1)} \prod_{j=1}^{k} F_{2 j-1}
$$

This completes the proof of (**).

## ON 0

H-205 Proposed by L. Carlitz, Duke University, Durham, North Carolina.
Evaluate the determinants of $n^{\text {th }}$ order

$$
\begin{aligned}
& D_{n}=\left|\begin{array}{cccccc}
z & 1 & & \\
-1 & q z & 1 & & & \\
& -1 & q^{2} z & 1 & & \\
\cdots & & & & -1 & q^{n-2} z \\
-1 & 1 \\
q^{n-1} z
\end{array}\right| \\
& \Delta_{n}=\left|\begin{array}{cccccc}
z & 1 & & & & \\
-1 & z & q & & & \\
& -1 & z & q^{2} & & \\
\cdots \cdots \cdots & \cdots & \cdots & & \\
& & & -1 & z & q^{n-2} \\
& & & & -1 & z
\end{array}\right|
\end{aligned}
$$

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.
If we expand the last row of each determinant by minors, we may readily obtain the following recursions:
(1)

$$
\begin{aligned}
& D_{n}=q^{n-1} z D_{n-1}+D_{n-2} \quad(n \geqslant 3) ; \quad D_{1}=z ; \quad D_{2}=q z^{2}+1 \\
& \Delta_{n}=z \Delta_{n-1}+q^{n-2} \Delta_{n-2} \quad(n \geqslant 3) ; \quad \Delta_{1}=z ; \quad \Delta_{2}=z^{2}+1
\end{aligned}
$$

(2)

The first recursion readily admits expression in continued fraction form. $D_{n}$ is equal to the numerator of the $n^{\text {th }}$ convergent of the simple continued fraction:

$$
z+1 / q z+1 / q^{2} z+1 / q^{3} z+\cdots
$$

An alternative notation for this infinite simple continued fraction is:

$$
\left[z, q z, q^{2} z, q^{3} z, \cdots, q^{n-1} z, \cdots\right]
$$

Recursion (2) may also be expressed in continued fraction form, but as it stands, it cannot be expressed in the form of a simple continued fraction, i.e., one with continued numerators of unity. If, however, we make the substitution:

$$
\begin{equation*}
\Delta_{n}=q^{1 / 2\left(n^{2}-2 n\right)} C_{n} \quad(n=1,2,3, \cdots) \tag{3}
\end{equation*}
$$

then (2) reduces to a form similar to that of (1), namely:

$$
\begin{equation*}
c_{n}=z q^{-1 / 4(2 n-3)} c_{n-1}+c_{n-2} \quad(n \geqslant 3) ; \quad c_{9}=z q^{1 / 4} ; \quad c_{2}=z^{2}+1 \tag{4}
\end{equation*}
$$

Thus, $C_{n}$ is equal to the numerator of the $n^{\text {th }}$ convergent of the simple continued fraction:

$$
\left[z q^{1 / 4}, z q^{-1 / 4}, z q^{-3 / 4}, \cdots, z q^{-1 / 4(2 n-3)}, \cdots\right]
$$

$\Delta_{n}$ is then found, by using (3).
Also solved by the Proposer.

## UNITY OF ROOTS

H-206 Proposed by P. Bruckman, University of Illinois, Urbana, Illinois.
Prove the identity:

$$
\frac{1}{1-x^{n}}=\frac{1}{n} \sum_{k=0}^{m-1} \frac{1}{1-x e^{2 k \pi i / n}}
$$

Solution by C. Bridger, Springfield, Illinois
Let $a, b, c, \cdots k, \cdots$ be the $n n^{t h}$ roots of unit. Among them, say the $k^{t h}$, is $e^{2 k \pi i / n}$. Put $x=1 / y$ and set $y^{n}-1=(y-a)(y-b)(y-c) \cdots(y-k) \cdots$. The logarithmic derivative is

$$
\frac{n y^{n-1}}{y^{n}-1}=\frac{1}{y-a}+\frac{1}{y-b}+\cdots+\frac{1}{y-k}+\cdots
$$

But this is exactly what the identity becomes when $x$ is replaced by $1 / y$ and the extra $y$ is discarded. The next and final step is to replace $y$ in the logarithmic derivative with $1 / x$, discard the extra $x$ and divide both sides by $n$.

Also solved by G. Lord and the Proposer.

