ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0, F_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$.

CORRECTED PROBLEM

B-279 Correction of typographical error in Vol. 12, No. 1 (February 1974).

Find a closed form for the coefficient of x^n in the Maclaurin series expansion of $(x+2x^2)/(1-x-x^2)^2$.

PROBLEMS PROPOSED IN THIS ISSUE

B-286 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let g be the "golden ratio" defined by $g = \lim_{n\to\infty} (F_n/F_{n+1})$. Simplify

$$\sum_{i=0}^{n} \binom{n}{i} g^{2n-3i} .$$

B-287 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let g be as in B-286. Simplify

$$g^2 \left\{ (-1)^{n-1} [F_{n-3} - gF_{n-2}] + g + 2 \right\} .$$

B-288 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove that $F_{2n(4k+1)} \equiv F_{2n} \pmod{L_{2n}}$ for all integers n and k.

B-289 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove that $F_{(2n+1)(2k+1)} \equiv F_{2n+1} \pmod{L_{2n+1}}$ for all integers n and k.

B-290 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Obtain a closed form for

$$2n + 1 + \sum_{k=1}^{n} (2n+1-2k)F_{2k} .$$

B-291 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the second-order recursion relation for $\{z_n\}$ given that

$$z_n = \sum_{k=0}^{n} \binom{n}{k} y_k \quad \text{and} \quad y_{n+2} = ay_{n+1} + by_n ,$$

where a and b are constants.

SOLUTIONS

LUCAS SUM MULTIPLES OF 5 AND 10

B-262 Proposed by Herta T. Freitag, Roanoke, Virginia.

(a) Prove that the sum of n consecutive Lucas numbers is divisible by 5 if and only if n is a multiple of 4.

(b) Determine the conditions under which a sum of n consecutive Lucas numbers is a multiple of 10.

Composite of Solutions by Graham Lord, Temple University, Philadelphia, Pennsylvania, and Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

The sum $S = L_{a+1} + L_{a+2} + \cdots + L_{a+n}$ of n consecutive Lucas numbers is equal to $L_{a+n+2} - L_{a+2}$; hence $d|s$ if and only if $L_{a+n+2} \equiv L_{a+2} \pmod d$.

(a) Modulo 5, the Lucas sequence is the block of four numbers 1, 3, 4, 2 repeated endlessly. Thus $5|S$ if and only if $4|n$.

(b) Modulo 10, the Lucas sequence is the block of twelve numbers

$$1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2$$

repeated endlessly. From this one sees that $10|S$ [or equivalently, $L_{a+n+2} \equiv L_{a+2} \pmod{10}$] if and only if either (i) $12|n$, or (ii) $12|(n-4)$ and $3|(a+1)$, or (iii) $12|(n-8)$ and $3|a$.

Also solved by C.B.A. Peck and the Proposer. Partial solutions were received from Paul S. Bruckman, Ralph Garfield, and David Zeitlin.

LUCASLIKE SEQUENCE

B-263 Proposed by Timothy B. Carroll, Graduate Student, Western Michigan University, Kalamazoo, Michigan.

Let $S_n = a^n + b^n + c^n + d^n$, where $a, b, c,$ and d are the roots of $x^4 - x^3 - 2x^2 + x + 1 = 0$.

(a) Find a recursion formula for S_n.

(b) Express S_n in terms of the Lucas number L_n.

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

(a) Since $a^4 - a^3 - 2a^2 + a + 1 = 0$, then, for $n = 0, 1, 2, \cdots$,

$$a^{n+4} - a^{n+3} - 2a^{n+2} + a^{n+1} + a^n = 0;$$

a similar relation holds for $b, c,$ and d. Adding these four equations, we obtain the recursion:

$$S_{n+4} - S_{n+3} - 2S_{n+2} + S_{n+1} + S_n = 0 \qquad (n = 0, 1, 2, \cdots)$$

(b) $$x^4 - x^3 - 2x^2 + x + 1 = (x^2 - 1)(x^2 - x - 1) = (x-1)(x+1)(x-\alpha)(x-\beta).$$

So

$$S_n = 1 + (-1)^n + \alpha^n + \beta^n = 1 + (-1)^n + L_n .$$

Also solved by Clyde A. Bridger, Herta T. Freitag, Ralph Garfield, Graham Lord, Jeffrey Shallit, Paul Smith, M.N.S. Swamy, Gregory Wulczyn, David Zeitlin, and the Proposer.

FIBONACCI PRODUCT

B-264 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Use the identities $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$ and $F_n^2 - F_{n-2}F_{n+2} = (-1)^n$ to obtain a factorization of $F_n^4 - 1$.

Solution by David Zeitlin, Minneapolis, Minnesota.

We note that

$$F_n^4 - 1 = \left\{ F_n^2 + (-1)^n \right\} \left\{ F_n^2 - (-1)^n \right\} = F_{n-1}F_{n+1}F_{n-2}F_{n+2} .$$

In the paper by D. Zeitlin, "Generating Functions for Products of Recursive Sequences," *Transactions of the Amer. Math. Soc.*, 116 (April, 1965), pp. 300-315, it was shown on p. 304 that if $H_{n+2} = H_{n+1} + H_n$, then for $n = 0, 1, \cdots,$

(1) $$H_{n-2}H_{n-1}H_{n+1}H_{n+2} = H_n^4 - (H_2^4 - H_0H_1H_3H_4) .$$

Thus, if $H_0 = 0$ and $H_1 = 1$, the $H_n = F_n$ and (1) gives the above result. If $H_0 = 2$ and $H_1 = 1$, then $H_n = L_n$ and (1) gives

(2) $$L_{n-2}L_{n-1}L_{n+1}L_{n+2} = L_n^4 - 25 \qquad (n = 0, 1, \cdots) .$$

Also solved by Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Ralph Garfield, Graham Lord, C.B.A. Peck, M.N.S. Swamy, Gregory Wulczyn, and the Proposer.

FIBONACCI NUMBERS FOR POWERS OF 3

B-265 Proposed by Zalman Usiskin, University of Chicago, Chicago, Illinois.

Let F_n and L_n be designated as $F(n)$ and $L(n)$. Prove that

$$F(3^n) = \prod_{k=0}^{n-1} [L(2\cdot 3^k) - 1] .$$

Composite of solutions by Ralph Garfield, College of Insurance, N.Y., N.Y., and David Zeitlin, Minneapolis, Minn.

Using the Binet formulas $F(n) = (a^n - b^n)/(a - b)$ and $L(n) = a^n + b^n$, one easily shows that

$$F(3m)/F(m) = L(2m) + (-1)^m .$$

This with $m = 3^k, 0 \leqslant k \leqslant n - 1$, and the facts that $F(1) = 1$ and 3^k is odd, help us obtain

$$F(3^n) = \prod_{k=0}^{n-1} \frac{F(3^{k+1})}{F(3^k)} = \prod_{k=0}^{n-1} [L(2\cdot 3^k) - 1] .$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, C.B.A. Peck, M.N.S. Swamy, Gregory Wulczyn, and the Proposer.

LUCAS NUMBERS FOR POWERS OF 3

B-266 Proposed by Zalman Usiskin, University of Chicago, Chicago, Illinois.

Let L_n be designated as $L(n)$. Prove that

$$L(3^n) = \prod_{k=0}^{n-1} [L(2\cdot 3^k) + 1] .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Since $L(3m) = L(m)[L(2m) - (-1)^m]$, we have, for $m = 3^k$, $0 \leqslant k \leqslant n - 1$,

$$L(3^n) = \prod_{k=0}^{n-1} \frac{L(3^{k+1})}{L(3^k)} = \prod_{k=0}^{n-1} [L(2\cdot 3^k) + 1].$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Graham Lord, C.B.A. Peck, M.N.S. Swamy, Gregory Wulczyn, and the Proposer.

REGULAR POLYGON RELATION

B-267 Proposed by Marjorie Bicknell, Wilcox High School, Santa Clara, California.

Let a regular pentagon of side p, a regular decagon of side d, and a regular hexagon of side h be inscribed in the same circle. Prove that these lengths could be used to form a right triangle; i.e., that $p^2 = d^2 + h^2$.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Hobson, in *Plane and Advanced Trigonometry,* on page 31 states:

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4}, \qquad \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

$$p = 2r \sin 36^\circ, \qquad h = r, \qquad d = 2r\sin 18^\circ$$

$$h^2 + d^2 = r^2 + \frac{4r^2}{16}(6 - 2\sqrt{5}) = \frac{(5-\sqrt{5})}{2} r^2$$

$$p^2 = \frac{4r^2}{16}(10 - 2\sqrt{5}) = \frac{5-\sqrt{5}}{2} r^2$$

$$\therefore p^2 = h^2 + d^2 .$$

Also solved by Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Graham Lord, C.B.A. Peck, Paul Smith, M.N.S. Swamy, David Zeitlin, and the Proposer.

*****

[Continued from Page 308.]

and in order for (ii) to be satisfied a must equal 1. Therefore, the given sequence must be the Fibonacci sequence.

NOTE: The most general sequence satisfying (i) has the form

$$\cdots, ax_1, x_1, x_0 = 0, x_1, ax_1, (a^2+1)x_1, \cdots.$$

Also, if condition (ii) is weakened to the restriction that two consecutive terms be relatively prime, then the most general sequence would have the form

$$\cdots, -a, 1, x_0 = 0, 1, a, a^2+1, \cdots$$

REFERENCE

1. V.E. Hoggatt, Jr., *Fibonacci and Lucas Numbers,* Houghton-Mifflin Co., New York, 1969.

****