THE DESIGN OF THE FOUR BINOMIAL IDENTITIES: MORIARTY INTERVENES

H. W. GOULD
West Virginia University, Morgantown, West Virginia 26506

We have seen in a previous episode [3] some of the artful disguises of the Moriarty identities. With skillful detective work we may unmask Moriarty in many situations. The case we are about to discuss arose in a study of questions asked me by David Zeitlin (personal correspondence of 9 August 1972), and reveals Moriarty in a fourfold fantasy; for there are actually a full dozen formulas to be analyzed. As corollaries we find other interesting sums. The objective in our study is pedagogical, viz. to show how to handle Moriarty. But let us hear Zeitlin's question.

"Are the following two related identities,

$$\sum_{k=j}^{m-1} \binom{k}{j}\binom{m+k}{2k+1}(-4)^{k-j} = (-1)^{m+j+1}\binom{m+j}{2j+1}, \tag{1}$$

$$\sum_{k=j}^{m-1} \binom{k}{j}\binom{m+k-1}{2k}(-4)^{k-j} = (-1)^{m+j+1}\binom{m+j}{2j+1}\frac{2m-1}{m+j} \tag{2}$$

listed (or special cases) in your tables [4]?" asked Zeitlin. "I am convinced that (1) and (2) are correct, but I am unable to prove it so."

Zeitlin stumbled onto these formulas as a consequence of several Fibonacci identities. Naturally no set of tables is ever complete; but the careful reader will ascertain at once that relation (2) is precisely (3.162) in my tables...precisely upon changing a few letters and shifting m to $m+1$. Relation (1) is not listed. However, relations (3.160) and (3.161) are obviously related to (1) and (2) in some manner, as we shall see.

We are therefore concerned at the outset with the four identities

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k} = (-1)^n \binom{n+a}{2a} 2^{2a}\frac{2n+1}{2a+1}, \qquad (3.162) \tag{3}$$

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k+1} 2^{2k} = (-1)^{n-1} \binom{n+a}{2a+1} 2^{2a}, \qquad (\text{Zeitlin}) \tag{4}$$

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k}\frac{n}{n+k} = (-1)^n \binom{n+a}{2a} 2^{2a}\frac{n}{n+a}, \qquad (3.160) \tag{5}$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k}\frac{2n+1}{2k+1} = (-1)^n \binom{n+a}{2a} 2^{2a}, \qquad (3.161). \tag{6}$$

Here a is a non-negative integer; the range of summation in each case may start with $k=a$ if one prefers, since $\binom{k}{a} = 0$ for $0 \leqslant k < a$. However, we state the four in a more elegant form as above.

Relations (3) and (6) are inverses of each other; this is so because of the easy and well known inversion principle that

$$\sum_{k=a}^{n} (-1)^k \binom{k}{a} f(k) = (-1)^n g(n)$$

if and only if

$$\sum_{k=a}^{n} (-1)^k \binom{k}{a} g(k) = (-1)^n f(n)\,.$$

Thus we have only to prove the one to obtain the other. Observe that (4) and (5) are self-inverses.

There are various ways to prove (3)–(6) directly; to this we shall give attention. But the main object of our work will be to show that these four sums are equivalent to the following four sums:

(7) $$\sum_{k=0}^{n} \binom{2n+1}{2k}\binom{k}{n-a} = \binom{n+a}{2a} 2^{2a}\,\frac{2n+1}{2a+1}\,,$$

(8) $$\sum_{k=0}^{n} \binom{2n}{2k+1}\binom{k}{n-1-a} = \binom{n+a}{2a+1} 2^{2a+1}\,,$$

(9) $$\sum_{k=0}^{n} \binom{2n}{2k}\binom{k}{n-a} = \binom{n+a}{2a} 2^{2a}\,\frac{n}{n+a}\,,$$

(10) $$\sum_{k=0}^{n} \binom{2n+1}{2k+1}\binom{k}{n-a} = \binom{n+a}{2a} 2^{2a}\,.$$

These are the *four* relations of Moriarty. The attentive reader of [3] may at first think we proved *two* relations, and indeed we did. They were:

(11) $$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\binom{k}{r} = 2^{n-2r-1}\binom{n-r}{r}\frac{n}{n-r}\,,$$ (3.120) in [4],

and

(12) $$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2k+1}\binom{k}{r} = 2^{n-2r}\binom{n-r}{r}\,,$$ (3.121) in [4].

To see how we get (7)–(10) from these, proceed as follows. In (11) put $2n+1$ for n, and recall that $[n+\tfrac{1}{2}] = n$. Replace r by $n-a$. The result is (7). In (12) put $2n-1$ for n, and note that $[n-\tfrac{1}{2}] = n-1$. Replace r by $n-a-1$. The result is (8). In (11), put $2n$ for n and replace r by $n-a$. The result is (9). Finally, in (12), put $2n$ for n and replace r by $n-a$. The result is (10).

What we have done above is reveal the fourfold design of the Moriarty identities. These formulas occur frequently in trigonometric identities.

We shall need the easy formula

(13) $$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+k}{r} = (-1)^n \binom{x}{r-n}$$

valid for all real x; this is formula (3.47) in [4] and can be proved from the Vandermonde convolution, for example.

To carry out the proofs that the fourfold Moriarty (7)–(10) imply and are implied by (3)–(6), we need to note the following four sums:

(14) $$\sum_{k=0}^{a} \binom{2n+1}{2k}\binom{n-k}{a-k} = \binom{n+a}{2a} 2^{2a},$$ (3.149 in [4],

(15) $$\sum_{k=0}^{a} \binom{2n}{2k+1}\binom{n-1-k}{a-k} = \binom{n+a}{2a+1} 2^{2a+1}$$ (3.158) in [4].

(16) $$\sum_{k=0}^{a} \binom{2n}{2k}\binom{n-k}{a-k} = \frac{n}{n+a}\binom{n+a}{2a} 2^{2a},$$ (3.26) in [4],

and

(17) $$\sum_{k=0}^{a} \binom{2n+1}{2k+1}\binom{n-k}{a-k} = \frac{2n+1}{2a+1}\binom{n+a}{2a} 2^{2a},$$ (3.27) in [4].

By the way, formula (3.157) in [4] is redundant, being equivalent to (3.158) by a simple change of variable.

Relations (14)–(17) may be proved directly as we could even prove the original (3)–(6). They occur quite naturally in work with trigonometric identities, and I first came on them some years ago while studying Bromwich [1] wherein they are implicit...some other time we may discuss this case. Note how (14)–(17) differ from the corresponding (7)–(10) in that *'k'* has been replaced by *'n – k'* in each case, or *'n – k – 1'* for the transition from (8) to (15). The relations (14)–(17) may be called another of Moriarty's disguises. The design of the four changes here. For proofs of (14)–(17), see [5].

PROOFS

We turn now to the proofs. To begin with, we show that (3) may be found from (7) using (14) and (13). Here are the step-by-step details:

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k} = \sum_{k=0}^{n} (-1)^k \binom{k}{a} \sum_{j=0}^{k} \binom{2n+1}{2j}\binom{n-j}{k-j}, \quad \text{by (14)},$$

$$= \sum_{j=0}^{n} \binom{2n+1}{2j} \sum_{k=j}^{n} (-1)^k \binom{k}{a}\binom{n-j}{k-j}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n+1}{2j} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k}\binom{k+j}{a}, \quad \text{by change of variable,}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n+1}{2j} (-1)^{n-j} \binom{j}{a-n+j}, \quad \text{by (13),}$$

$$= (-1)^n \sum_{j=0}^{n} \binom{2n+1}{2j}\binom{j}{n-a} = (-1)^n \binom{n+a}{2a} 2^{2a} \frac{2n+1}{2a+1}, \quad \text{by (7).}$$

The proofs that (8) and (15) imply (4), that (9) and (16) imply (5), and that (10) and (17) imply (6) are done in similar fashion, using (13), and we give the details so the reader will have no mystery left to solve.

The steps may be reversed so that (7) follows from (3) using (14) and (13), etc., so that we find relations (3)–(6) equivalent to relations (7)–(10) assuming relations (14)–(17).

To show that (4) may be found from (8) using (15) and (13):

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k+1} 2^{2k+1} = \sum_{k=0}^{n} (-1)^k \binom{k}{a} \sum_{j=0}^{k} \binom{2n}{2j+1}\binom{n-1-j}{k-j}, \quad \text{by (15)}$$

$$= \sum_{j=0}^{n} \binom{2n}{2j+1} \sum_{k=j}^{n} (-1)^k \binom{k}{a}\binom{n-1-j}{k-j}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n}{2j+1} \sum_{k=0}^{n-j} (-1)^k \binom{k+j}{a} \binom{n-1-j}{k}, \qquad \text{by change of variable,}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n}{2j+1} \sum_{k=0}^{n-j-1} (-1)^k \binom{n-1-j}{k} \binom{k+j}{a},$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n}{2j+1} (-1)^{n-j-1} \binom{j}{a-(n-j-1)}, \qquad \text{by (13),}$$

$$= (-1)^{n-1} \sum_{j=0}^{n} \binom{2n}{2j+1} \binom{j}{a-n+j+1}$$

$$= (-1)^{n-1} \sum_{j=0}^{n} \binom{2n}{2j+1} \binom{j}{n-1-a}$$

$$= (-1)^{n-1} \binom{n+a}{2a+1} 2^{2a+1}, \qquad \text{by (8).}$$

To show that (5) may be found from (9) using (16) and (13):

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a} \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} = \sum_{k=0}^{n} (-1)^k \binom{k}{a} \sum_{j=0}^{k} \binom{2n}{2j} \binom{n-j}{k-j}, \qquad \text{by (16),}$$

$$= \sum_{j=0}^{n} \binom{2n}{2j} \sum_{k=j}^{n} (-1)^k \binom{k}{a} \binom{n-j}{k-j}$$

$$= \sum_{j=0}^{n} \binom{2n}{2j} (-1)^j \sum_{k=0}^{n-j} (-1)^k \binom{k+j}{a} \binom{n-j}{k}, \qquad \text{by change of variable,}$$

$$= \sum_{j=0}^{n} (-1)^j \binom{2n}{2j} (-1)^{n-j} \binom{j}{a-n+j}, \qquad \text{by (13),}$$

$$= (-1)^n \sum_{j=0}^{n} \binom{2n}{2j} \binom{j}{a-n+j} = (-1)^n \sum_{j=0}^{n} \binom{2n}{2j} \binom{j}{n-a}$$

$$= (-1)^n \frac{n}{n+a} \binom{n+a}{2a} 2^{2a}, \qquad \text{by (9).}$$

To show that (6) may be found from (10) using (17) and (13):

$$\sum_{k=0}^{n} (-1)^k \binom{k}{a} \frac{2n+1}{2k+1} \binom{n+k}{2k} 2^{2k} = \sum_{k=0}^{n} (-1)^k \binom{k}{a} \sum_{j=0}^{k} \binom{2n+1}{2j+1} \binom{n-j}{k-j}, \qquad \text{by (17),}$$

$$= \sum_{j=0}^{n} \binom{2n+1}{2j+1} \sum_{k=j}^{n} (-1)^k \binom{k}{a} \binom{n-j}{k-j},$$

$$= \sum_{j=0}^{n} \binom{2n+1}{2j+1} (-1)^n \binom{j}{n-a}, \qquad \text{by (13),}$$

$$= (-1)^n \sum_{j=0}^{n} \binom{2n+1}{2j+1}\binom{j}{n-a} = (-1)^n \binom{n+a}{2a} 2^{2a}, \qquad \text{by (10).}$$

PROOFS USING GENERATING FUNCTIONS

From the binomial theorem we have

$$\sum_{n=0}^{\infty} \binom{n+a}{a} x^n = (1-x)^{-a-1} \quad \text{or} \quad \sum_{n=a}^{\infty} \binom{n}{a} x^n = x^a(1-x)^{-a-1}.$$

In particular

$$\sum_{n=a+1}^{\infty} \binom{n+a}{2a+1} x^n = x^{a+1}(1-x)^{-2a-2}. \tag{18}$$

We first use (18) to prove (4) of Zeitlin, as follows:

$$\begin{aligned}
\sum_{n=a}^{\infty} t^n \sum_{k=a}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k+1} 2^{2k} &= \sum_{k=a}^{\infty} (-1)^k \binom{k}{a} 2^{2k} \sum_{n=k}^{\infty} t^n \binom{n+k}{2k+1} \\
&= \sum_{k=a}^{\infty} (-1)^k \binom{k}{a} 2^{2k} \sum_{n=k+1}^{\infty} t^n \binom{n+k}{2k+1} \\
&= \sum_{k=a}^{\infty} (-1)^k \binom{k}{a} 2^{2k} t^{k+1}(1-t)^{-2k-2} \\
&= \frac{t}{(1-t)^2} \sum_{k=a}^{\infty} \binom{k}{a} \left\{\frac{-4t}{(1-t)^2}\right\}^k \\
&= \frac{t}{(1-t)^2} \left\{\frac{-4t}{(1-t)^2}\right\}^a \left\{1 + \frac{4t}{(1-t)^2}\right\}^{-a-1} \\
&= (-1)^a 2^{2a} t^{a+1}(1+t)^{-2a-2}.
\end{aligned}$$

But also

$$-2^{2a} \sum_{n=a+1}^{\infty} \binom{n+a}{2a+1} (-t)^n = -2^{2a} t^{a+1}(-1)^{a+1}(1+t)^{-2a-2},$$

so that each side of (4) gives the same generating function, whence, by uniqueness of the expansion, (4) is proved.

The generating function for (3) is similar, and is in fact

$$(-1)^a 2^{2a} t^a (1-t)(1+t)^{-2a-2}.$$

We have on the one hand

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} (-1)^k \binom{k}{a}\binom{n+k}{2k} 2^{2k} &= \sum_{k=0}^{\infty} (-1)^k \binom{k}{a} 2^{2k} \sum_{n=k}^{\infty} \binom{n+k}{2k} t^n \\
&= \sum_{k=a}^{\infty} (-1)^k \binom{k}{a} 2^{2k} t^k \sum_{n=0}^{\infty} \binom{n+2k}{2k} t^n
\end{aligned}$$

$$= \sum_{k=a}^{\infty} (-1)^k \binom{k}{a} 2^{2k} t^k (1-t)^{-2k-1} = \frac{1}{1-t} \sum_{k=a}^{\infty} \binom{k}{a} \left\{ \frac{-4t}{(1-t)^2} \right\}^k = (-1)^a \frac{2^{2a} t^a (1-t)}{(1+t)^{2a+2}} .$$

On the other hand

$$(-1)^a \frac{2^{2a} t^a (1-t)}{(1+t)^{2a+2}} = (-1)^a 2^{2a} t^a (1-t) \sum_{n=0}^{\infty} \binom{n+2a+1}{2a+1} (-t)^n = 2^{2a}(1-t) \sum_{n=a}^{\infty} \binom{n+a+1}{2a+1} (-t)^n$$

$$= 2^{2a} \sum_{n=a}^{\infty} \binom{n+a+1}{2a+1} (-t)^n - 2^{2a} \sum_{n=a}^{\infty} \binom{n+a+1}{2a+1} (-1)^n t^{n+1}$$

$$= 2^{2a} \sum_{n=a}^{\infty} \binom{n+a+1}{2a+1} (-t)^n + 2^{2a} \sum_{n=a+1}^{\infty} \binom{n+a}{2a+1} (-t)^n$$

$$= 2^{2a} \sum_{n=a}^{\infty} \left\{ \binom{n+a+1}{2a+1} + \binom{n+a}{2a+1} \right\} (-t)^n = 2^{2a} \sum_{n=a}^{\infty} (-t)^n \binom{n+a}{2a+1} \frac{2n+1}{2a+1} ,$$

so that (3) is proved.

PROOFS USING HYPERGEOMETRIC FUNCTIONS

The ordinary hypergeometric function is given by

$$F(a,b;c;x) = \sum_{k=0}^{\infty} (-1)^k \binom{-a}{k} \binom{-b}{k} \binom{-c}{k}^{-1} x^k . \tag{19}$$

Since it is easy to verify that

$$\binom{n+k}{2k} 2^{2k} = \binom{n}{k} \binom{-n-1}{k} \binom{-\frac{1}{2}}{k}^{-1} , \tag{20}$$

it is easy to see that series (3) may be put in hypergeometric form using a^{th} derivatives; in fact because

$$D_x^a x^k = a! \binom{k}{a} x^{k-a} ,$$

$$D_x^a F(-n, n+1; \tfrac{1}{2}; x)\Big|_{x=1} = a! \sum_{k=0}^{n} (-1)^k \binom{k}{a} \binom{n+k}{2k} 2^{2k} = a! S .$$

Now a standard result about the hypergeometric function is that

$$D_x^m F(a,b;c;x) = m! \binom{a+m-1}{m} \binom{b+m-1}{m} \binom{c+m-1}{m}^{-1} F(a+m, b+m; c+m; x) ,$$

and thus

$$a!S = a! \binom{-n+a-1}{a} \binom{n+a}{a} \binom{\frac{1}{2}+a-1}{a}^{-1} F(-n+a, n+1+a; \tfrac{1}{2}+a; 1)$$

$$= a! \binom{n}{a} \binom{n+a}{a} \binom{-\frac{1}{2}}{a}^{-1} \frac{(-\frac{1}{2}+a)!(-3/2-a)!}{(-\frac{1}{2}+n)!(-3/2-n)!} ,$$

by Gauss' formula for a terminating $F(-m, b; c; 1)$, since $a \leq n$,

$$= (-1)^a 2^{2a} \frac{(n+a)!(-\frac{1}{2}+a)!(-3/2-a)!a!}{(n-a)!(-\frac{1}{2}+n)!(-3/2-n)!(2a)!}$$

$$= (-1)^a 2^{2a} \frac{(n+a)!(-\frac{1}{2}+a)!(-\frac{1}{2}-a)!a!}{(n-a)!(-\frac{1}{2}+n)!(-\frac{1}{2}-n)!(2a)!} \cdot \frac{2n+1}{2a+1} .$$

Making use of the formula $(-\frac{1}{2}+m)!(-\frac{1}{2}-m)! = (-1)^m \pi$, this then reduces to

$$(-1)^n \frac{(n+a)!(2n+1)a!}{(n-a)!(2a+1)!} 2^{2a} ,$$

which proves (3).

Somewhat similar proofs may be given for (4)–(6). Because

$$(21)\qquad \binom{n+k}{2k+1} 2^{2k} = \frac{n}{2k+1}\binom{n-1}{k}\binom{-n-1}{k}\binom{-\frac{1}{2}}{k}^{-1},$$

some proofs of relations like (4) using hypergeometric series will involve integration techniques as well.

OTHER PROOFS BY DIFFERENTIATION

For any function f we have trivially

$$(22)\qquad \sum_{k=0}^{n}\binom{n+k}{2k} f(k) = \sum_{k=0}^{n}\binom{n+k}{n-k} f(k) = \sum_{k=0}^{n}\binom{2n-k}{k} f(n-k).$$

Thus, for example,

$$(23)\qquad a!\sum_{k=0}^{n}\binom{n+k}{2k}\binom{k}{a}x^{k-a} = a!\sum_{k=0}^{n}\binom{2n-k}{k}\binom{n-k}{a}x^{n-k-a} = D_x^a\sum_{k=0}^{n}\binom{2n-k}{k}x^{n-k}.$$

The series

$$\sum_{k=0}^{[n/2]}\binom{n-k}{k}z^k$$

can be written in a (complicated) closed form. See relation (1.70)–(1.71) in [4]. In principle then, one can obtain (23) in closed form. The form of the series again shows how our work is related to Fibonacci numbers since we know that

$$\sum_{k=0}^{[n/2]}\binom{n-k}{k} = F_{n+1} = F_n + F_{n-1}, \qquad F_0 = 0,\quad F_1 = 1.$$

A RECURRENCE RELATION

Some other interesting things can be deduced by looking briefly at a recurrence relation for (4). Since

$$\binom{n+k}{2k} + \binom{n+k}{2k+1} = \binom{n+k+1}{2k+1},$$

we find easily

$$\sum_{k=0}^{n}(-1)^k\binom{k}{a}\binom{n+k}{2k}2^{2k} + \sum_{k=0}^{n}(-1)^k\binom{k}{a}\binom{n+k}{2k+1}2^{2k} = \sum_{k=0}^{n}(-1)^k\binom{k}{a}\binom{n+k+1}{2k+1}2^{2k}$$

or, in virtue of (3), then

$$(24)\qquad S_{n+1} - S_n = (-1)^n\binom{n+a}{2a}2^{2a}\frac{2n+1}{2a+1},$$

where S_n is Zeitlin's series in (4).

Recalling that

$$\sum_{j=0}^{n-1}(S_{j+1} - S_j) = S_n - S_0,$$

we next find, since $S_0 = 0$, that for arbitrary S_j,

$$S_n = \frac{2^{2a}}{2a+1}\sum_{j=0}^{n-1}(-1)^j\binom{j+a}{2a}(2j+1).$$

and unless we know how to sum this in closed form the method yields nothing. But since we do know the value of S_n, we may look on this as a way to have evaluated a new series, and so we have found in fact

$$(25)\qquad \sum_{k=0}^{n}(-1)^k\binom{k+a}{2a}(2k+1) = (-1)^n(n+a+1)\binom{n+a}{2a}.$$

INVERSION

As the reader will recall from the previous Moriarty episode [3], a good detective learns something by adroit use of inversion. Indeed, we now make use of the following inversion principle, that

$$f(n) = \sum_{k=0}^{n} \binom{n+k}{2k} g(k)$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{2n+1}{n-k} \frac{2k+1}{2n+1} f(k) .$$

This is relation (21) on p. 67 of [6]. Applying this principle to (3), we find by inversion that

$$\sum_{k=0}^{n} \binom{2n+1}{n-k} \binom{k+a}{2a} (2k+1)^2 = (2n+1)(2a+1) \binom{n}{a} 2^{2n-2a} . \tag{26}$$

This relation might be somewhat difficult to come by without the inversion application and may possibly serve in some way to indicate the fondness with which I like to use inversion techniques to establish new identities.

Riordan gives another inversion formula, same page, which is

$$f(n) = \sum_{k=0}^{n} \frac{2n}{n+k} \binom{n+k}{2k} g(n-k)$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{n-k} f(k) .$$

This may be used to obtain other interesting series.

A FINAL REMARK

The four series (14)–(17) were posed as a problem in the *American Math. Monthly* [5] and the solution by M.T.L. Bizley used just simple coefficient comparison in suitable generating functions. We asked there to sum

$$\sum_{k=0}^{n} \binom{2x+i}{2k+j} \binom{x-k}{n-k} \tag{27}$$

for all real x and for $i = 0, 1,$ and $j = 0, 1.$ Our question as to whether the series can be summed for *all* integers i, j remains unanswered.

It seems of value to remark also that in the case of (16) and (17) we have factorizations that are of interest:

$$\sum_{k=0}^{n} \binom{2x}{2k} \binom{x-k}{n-k} = \frac{2^{2n}}{(2n)!} \prod_{k=0}^{n-1} (x^2 - k^2), \tag{28}$$

and

$$\sum_{k=0}^{n} \binom{2x+1}{2k+1} \binom{x-k}{n-k} = \frac{2x+1}{(2n+1)!} \prod_{k=0}^{n-1} \left\{ (2x+1)^2 - (2k+1)^2 \right\} . \tag{29}$$

We leave it as an exercise for the reader to determine whether factorizations exist for (14) and (15). This has an easy affirmative answer.

Sherlock Holmes [2, p. 470] remarked about the original Professor Moriarty that "the man pervades London, and no one has heard of him...I tell you Watson, in all seriousness, that if I could beat that man, if I could free society of him, I should feel that my own career had reached its summit, and I should be prepared to turn to some more placid line in life." Our mathematical Moriarty formulas pervade mathematics and his formulas are the secret behind half of the conspiracy of formulas we meet with in our work. Moriarty is everywhere Watson, everywhere! Look closely and you cannot help seeing him and his formulas.

EPILOGUE

As if to show the force of the remark that Moriarty is everywhere, if we just look for him, it is instructive to say now that relations (14)–(17) are nothing in the world but relations (7)–(10) of Moriarty viewed in a slightly different way. An easy way to see this is to make sufficient use of the following simple operations on series and binomial coefficients:

$$\binom{m}{k} = 0$$

for $k > m$, and, typically,

$$\binom{n-k}{a-k} = \binom{n-k}{n-a}, \quad \binom{2n}{2k} = \binom{2n}{2n-2k}, \quad \sum_{k=0}^{n} f(k) = \sum_{k=0}^{n} f(n-k).$$

Illustration. We show that (14) is equivalent to (10):

$$\sum_{k=0}^{a} \binom{2n+1}{2k}\binom{n-k}{a-k} = \sum_{k=0}^{a} \binom{2n+1}{2k}\binom{n-k}{n-a} = \sum_{k=0}^{n} \binom{2k+1}{2k}\binom{n-k}{n-a}$$

$$= \sum_{k=0}^{n} \binom{2n+1}{2n-2k}\binom{k}{n-a} = \sum_{k=0}^{n} \binom{2n+1}{2k+1}\binom{k}{n-a} .$$

Similarly (15) is equivalent to (8):

$$\sum_{k=0}^{a} \binom{2n}{2k+1}\binom{n-1-k}{a-k} = \sum_{k=0}^{a} \binom{2n}{2k+1}\binom{n-1-k}{n-1-a} = \sum_{k=0}^{n-1} \binom{2n}{2k+1}\binom{n-1-k}{n-1-a}$$

$$= \sum_{k=0}^{n-1} \binom{2n}{2n-2k-1}\binom{k}{n-1-a} = \sum_{k=0}^{n-1} \binom{2n}{2k+1}\binom{k}{n-1-a} .$$

The reader should now have no difficulty in showing that (16) is equivalent to (9), and that (17) is equivalent to (7). The equivalences are so complete and obvious that we wonder how anyone could miss them. Thus we have used the Moriarty formulas twice in our proofs of (3)–(6). Moriarty, Moriarty, all is Moriarty! "Indubitably my Dear Watson, indubitably."

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Theorem. The Fibonacci numbers form the only sequence of integers for which its extended sequence satisfies:

(i) $$x_{-n} = (-1)^{n+1}x_n, \quad n \text{ an integer},$$

(ii) any three consecutive terms of the sequence are relatively prime.

Proof. Let x_n be a sequence which satisfies (i) and (ii); then,

$$x_1 = ax_0 + bx_{-1} = ax_0 + bx_1 .$$

Hence,

(*) $$ax_0 = (1-b)x_1 .$$

Now,

$$x_0 = ax_{-1} + bx_{-2} = ax_1 - b(ax_1 + bx_0),$$

which implies that

$$(1+b^2)x_0 = ax_1(1-b) = a^2x_0 ,$$

using (*). Since the sequence is nontrivial x_0 and x_1 cannot both be 0. If $x_0 \neq 0$; then $a^2 = 1 + b^2$, which implies that $a = \pm 1$ and $b = 0$. In either of these cases, (ii) will not hold. Hence, $x_0 = 0$. From (*) it follows that $b = 1$.

Thus far, the sequence hs the form $x_0 = 0, x_1, ax_1, \cdots$; hence, in order to satisfy (ii), x_1 must equal 1. This yields a sequence of the form

$$x_0 = 0, 1, a, a^2+1, a^3+2a, \cdots$$

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