ON POLYNOMIALS RELATED TO TCHEBICHEF POLYNOMIALS OF THE SECOND KIND

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1. Tchebichef polynomials of the second kind have been defined by

$$U_{n+1}(x) = 2x\,U_n(x) - U_{n-1}(x)\,,$$
$$U_0 = 1, \quad U_1 = 2x\,.$$

It is known [1] that

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}\,,$$

and

$$U_n(x) = \sum_{r=0}^{[n/2]} \binom{n-r}{r} (-1)^r (2x)^{n-2r}\,.$$

Also [2]

$$F_{n+1} = i^{-n}\,U_n(i/2)\,,$$

where F_n represents the n^{th} Fibonacci number.

The first few polynomials are

$$U_0(x) = 1$$
$$U_1(x) = 2x$$
$$U_2(x) = 4x^2 - 1$$
$$U_3(x) = 8x^3 - 4x$$
$$U_4(x) = 16x^4 - 12x^2 + 1\,.$$

Figure 1

If we take the sums along the rising diagonals in the expression on the right-hand side, we obtain an interesting polynomial $p_n(x)$, which is closely related to Fibonacci numbers.

The first few polynomials are

$$(1.1) \qquad p_1(x) = 1, \quad p_2(x) = 2x, \quad p_3(x) = 4x^2\,, \quad p_4(x) = 8x^3 - 1, \quad p_5(x) = 16x^4 - 4x\,.$$

In this note we shall derive the generating function, recurrence relation and a few interesting properties of these polynomials.

2. On putting $2x = y$ in the expansion on the right-hand side in Figure 1 we obtain

Columns

Rows	0	1	2	3
1	-1			
2	y			
3	y^2	-1		
4	y^3	$-2y$		
5	y^4	$-3y^2$	1	
	⋮			

Figure 2

The generating function for the k^{th} column in Figure 2 is $(-1)^k(1-ty)^{-(k+1)}$. Since we are summing along the rising diagonals, the row adjusted generating function for the k^{th} column becomes

$$h_k(y) \equiv (-1)^k(1-ty)^{-(k+1)}t^{3k+1} .$$

Since

$$\sum_{k=0}^{\infty} h_k(y) = \frac{1}{1-ty} \sum_{k=0}^{\infty} \left(\frac{-t^3}{1-ty}\right)^k$$

$$= \frac{t}{1-ty+t^3} ,$$

we have

$$\text{(2.1)} \qquad G(x,t) = \sum_{n=0}^{\infty} p_n(x)t^n = \frac{t}{1-2xt+t^3} .$$

From (2.1) we obtain

$$\sum_{n=1}^{\infty} p_n(x)t^n = t(1-2xt+t^3)^{-1}$$

On expanding the right-hand side and comparing the coefficients of t^{n+1}, we obtain

$$\text{(2.2)} \qquad p_{n+1}(x) = (2x)^n - \binom{n-2}{1}(2x)^{n-3} + \binom{n-4}{2}(2x)^{n-6} + \cdots = \sum_{r=0}^{[n/3]} \binom{n-2r}{r}(-1)^r(2x)^{n-3r} .$$

Again from (2.1) we have

$$(1-2xt+t^3)\sum_{n=1}^{\infty} p_n(x)t^n = t .$$

On equating coefficient of t^{n+3} on both sides, we obtain the recurrence relation

$$\text{(2.3)} \qquad p_{n+3}(x) = 2xp_{n+2}(x) - p_n(x), \quad n > 1, \quad p_1(x) = 1, \quad p_2(x) = 2x, \quad p_3(x) = 4x^2 .$$

Extending (2.3) we find that $p_o(x) = 0$.

From (2.1) we have

$$\text{(2.4)} \qquad G(x,t) = tF(2xt-t^3), \quad F(u) = (1-u)^{-1} .$$

Differentiating (2.4) partially with respect to x and t, we find that $G(x,t)$ satisfies the partial differential equation

$$2t\frac{\partial G}{\partial t} - (2x-3t^2)\frac{\partial G}{\partial x} - 2G = 0 .$$

Since

$$\frac{\partial G}{\partial t} = \sum_{n=1}^{\infty} np_n(x)t^{n-1}, \quad \frac{\partial G}{\partial x} = \sum_{n=1}^{\infty} p'_n(x)t^n ,$$

it follows that

$$2xp'_{n+2}(x) - 3p'_n(x) = 2(n+1)p_{n+2}(x) . \tag{2.5}$$

3. On substituting $x = 1$ in the polynomials $p_n(x)$, we obtain the sequence $\{P_n\}$ which has a recurrence relation

$$P_{n+2} = P_{n+1} + P_n + 1, \quad P_0 = 0. \quad P_1 = 1. \tag{3.1}$$

The sequence $\{P_n\}$ is related to the Fibonacci sequence $\{F_n\}$ by the relation

$$P_n - P_{n-1} = F_n ,$$

which leads to

$$P_n = \sum_{k=0}^{n} F_k . \tag{3.4}$$

From (3.4) several interesting properties of the sequence $\{P_n\}$ can be derived. A few of them are

(3.5)

(1) $$P_n = F_{n+2} - 1$$

(2) $$\sum_{k=1}^{n} P_k = F_{n+4} - (n+3)$$

(3) $$\sum_{k=1}^{n} P_k^2 = F_{n+2}F_{n+3} - 2F_{n+4} + (n+4)$$

(4) with $$\prod_{i=1}^{n} (1 + x^{L_i}) = a_0 a_1 x + \cdots + a_m x^m, \quad m = L_1 + L_2 + \cdots + L_n .$$

and q_n equal to the number of integers k such that both $0 < k < m$ and $a_k = 0$, Leonard [3] has proposed a problem to find a recurrence relation for q_n. The author [4] has shown that the recurrence relation is

$$q_{n+2} = q_{n+1} + q_n + 1, \quad q_1 = 0, \quad q_2 = 1 .$$

Comparing this result with (3.1) we observe that

$$P_n = q_{n+1} .$$

On using (3.5)—(1) and (2.2) we obtain

$$F_{n+3} = 1 + \sum_{r=0}^{[n/3]} \binom{n-2r}{r} (-1)^r 2^{n-3r}, \quad n \geq 0 , \tag{3.6}$$

a result which is believed to be undiscovered so far.

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4. Problem B-151, Solution submitted by D. V. Jaiswal.

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