ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-252 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let $A_{n\times n}$ be an $n\times n$ lower semi-matrix and $B_{n\times n}$, $C_{n\times n}$ be matrices such that $A_{n\times n}B_{n\times n} = C_{n\times n}$. Let $A_{k\times k}$, $B_{k\times k}$, $C_{k\times k}$ be the $k\times k$ upper left submatrices of $A_{n\times n}$, $B_{n\times n}$, and $C_{n\times n}$. Show $A_{k\times k}B_{k\times k} = C_{k\times k}$ for $k = 1, 2, \cdots, n$.

H-253 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{t=0}^{k}\binom{(\beta-1)n+t+1}{t}\sum_{j=0}^{n-k-1}\binom{n-k-1}{j}\sum_{m=0}^{j}(-1)^{n+m+k+1}\binom{j}{m}$$

$$\cdot\sum_{r=0}^{n+m-t-j-1}\binom{j}{n+m-j-t-r-1}\binom{2j+r-1}{r} = 2^{n-k-1}\binom{\beta n}{k},$$

where β is an arbitrary complex number and n and k are positive integers, $k < n$.

This identity, in the case $\beta = 2$, arose in solving a certain combinatorial problem in two different ways.

H-254 Proposed by R. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Consider the Fibonacci-Pascal Type Triangle given below.

$$\begin{array}{ccccccccc}
 & & & & F_1 & & & & \\
 & & & F_1 & & F_1 & & & \\
 & & F_1 & & F_2 & & F_1 & & \\
 & F_1 & & F_3 & & F_3 & & F_1 & \\
F_1 & & F_4 & & F_6 & & F_4 & & F_1 \\
 & & & & \cdots & & & &
\end{array}$$

Find a formula for the row sums of this array.

SOLUTIONS

ENUMERATION

H-226 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.

(i) Let k be a fixed positive integer. Find the number of sequences of integers $(a_1, a_2, \cdots, a_n)$ such that

$$0 \leqslant a_i \leqslant k \qquad (i = 1, 2, \cdots, n)$$

and if $a_i > 0$ then $a_i \neq a_{i-1}$ for $i = 2, \cdots, n$.

(ii) Let k be a fixed positive integer. Find the number of sequences of integers $(a_1, a_2, \cdots, a_n)$ such that

$$0 \leqslant a_i \leqslant k \qquad (i = 1, 2, \cdots, n)$$

and if $a_i > 0$ then $a_i \neq a_{i-1}$ for $i = 2, \cdots, n$; moreover $a_i = 0$ for exactly r values of i.

Solution by the Proposers.

Part (i).

Let $f_j(n)$ denote the number of such sequences with $a_n = j$. Then clearly

$$f_1(n) = \cdots = f_k(n)$$

and

$$\begin{cases} f_0(n+1) = f_0(n) + kf_1(n) \\ f_1(n+1) = f_0(n) + (k-1)f_1(n). \end{cases}$$

It follows that

$$(*) \qquad \begin{cases} f_0(n+2) = kf_0(n+1) + f_0(n) \\ f_1(n+2) = kf_1(n+1) + f_1(n). \end{cases}$$

Also

$$f_0(1) = 1, \qquad f_0(2) = k+1, \qquad f_1(1) = 1, \qquad f_1(2) = k.$$

It is convenient to take

$$f_0(0) = 0, \qquad f_1(0) = 1;$$

then (*) holds for all $n \geqslant 0$.

We now take

$$F_0(x) = \sum_{n=0}^{\infty} f_0(n)x^n = 1 + x + x^2 \sum_{0}^{\infty} (kf_0(n+1) + f_0(n))x$$

$$= 1 - (k-1)x + (kx + x^2) \sum_{0}^{\infty} f_0(n)x^n,$$

so that

$$F_0(x) = \frac{1-(k-1)x}{1-kx-x^2}.$$

Similarly

$$F_1(x) = \sum_{n=0}^{\infty} f_1(n)x^n = x + \sum_{0}^{\infty} (kf_1(n+1) + f_1(n))x^n,$$

which yields

$$F_1(x) = \frac{x}{1-kx-x^2}.$$

Let $S(n) = f_0(n) + kf_1(n)$ denote the total number of sequences satisfying the stated conditions. Then

$$\sum_{0}^{\infty} S(n)x^n = \frac{1+x}{1-kx-x^2}.$$

Since

$$\frac{1}{1-kx-x^2} = \sum_{r=0}^{\infty} x^r(k+x)^r = \sum_{r=0}^{\infty} x^r \sum_{j=0}^{r} \binom{r}{j} k^{r-j}x^j = \sum_{n=0}^{\infty} x^n \sum_{2j \leqslant n} \binom{n-j}{j} k^{n-2j},$$

it follows that

$$S(n) = \sum_{2j\leqslant n} \binom{n-j}{j} k^{n-2j} + \sum_{2j<n} \binom{n-j-1}{j} k^{n-2j-1}$$

Part (ii)

Let $f_j(n,r)$ denote the number of such sequence with $a_n = j$. Then clearly

$$f_1(n,r) = \cdots = f_k(n,r)$$

and

$$\begin{cases} f_0(n+1,r) = f_0(n,r-1) + kf_1(n,r-1) \\ f_1(n+1,r) = f_0(n,r) + (k-1)f_1(n,r) \end{cases} \qquad (n \geqslant 1,\ r \geqslant 0),$$

where $f_i(n,-1) = 0$.

Clearly

$$f_0(1,r) = \begin{cases} 1 & (r=1) \\ 0 & \text{(otherwise)} \end{cases}$$

$$f_1(1,r) = \begin{cases} 1 & (r=1) \\ 0 & \text{(otherwise)}. \end{cases}$$

Put

$$F_i(z,x) = \sum_{n=1}^{\infty} \sum_{r=0}^{n} f_i(n,r)z^n x^r \qquad (i = 1,2).$$

Then

$$F_0(z,x) = zx + \sum_{n=1}^{\infty} \sum_{r=1}^{n+1} (f_0(n,r-1) + kf_1(n,r-1))z^{n+1}x^r = zx + zxF_0(z,x) + kzxF_1(z,x),$$

$$F_1(z,x) = z + \sum_{n=1}^{\infty} \sum_{r=0}^{n} (f_0(n,r) + (k-1)f_1(n,r))z^{n+1}x^r = z + zF_0(z,x) + (k-1)zF_1(z,x).$$

Thus

$$\begin{cases} (1-zx)F_0(z,x) - kzxF_1(z,x) = zx \\ -zF_0(z,x) + (1-(k-1)z)F_1(z,x) = z. \end{cases}$$

It follows that

$$(*) \qquad \begin{cases} F_0(z,x) = \dfrac{zx + z^2x}{1-(x+k-1)z - z^2x} \\[2ex] F_1(z,x) = \dfrac{z}{1-(x+k-1)z-z^2x} \end{cases}$$

We have

$$\frac{1}{1-(x+k-1)z-z^2x} = \sum_{n=0}^{\infty} z^n(x+k-1+zx)^n = \sum_{j,s=0}^{\infty} \binom{j+s}{s} (x+k-1)^j x^s z^{j+2s}$$

$$= \sum_{n=0}^{\infty} z^n \sum_{2s\leqslant n} \binom{n-s}{s} (x+k-1)^{n-2s}x^s.$$

Hence

$$\sum_{r=0}^{n+1} f_1(n+1,r)x^r = \sum_{2s\leqslant n} \binom{n-s}{s} x^s(x+k-1)^{n-2s} = \sum_{2s\leqslant n} \binom{n-s}{s} x^s \sum_{t=0}^{n-2s} \binom{n-2s}{t} (k-1)^{n-2s-t}x^t$$

$$= \sum_{r=0}^{n} x^r \sum_{s=0}^{r} \binom{n-s}{s}\binom{n-2s}{r-s} (k-1)^{n-r-s},$$

so that

$$f_1(n+1,r) = \sum_{s=0}^{r} \binom{n-s}{s}\binom{n-2s}{r-s}(k-1)^{n-r-s}.$$

It is evident from (*) that

$$f_0(n+1,r+1) = f_1(n+1,r) + f_1(n,r-1).$$

Thus

$$f_0(n+1,r+1) = \sum_{s=0}^{r} \binom{n-s}{s}\binom{n-2s}{r-s}(k-1)^{n-r-s} + \sum_{s=0}^{r-1} \binom{n-s-1}{s}\binom{n-2s-1}{r-s-1}(k-1)^{n-r-s}.$$

Let $S(n,r) = f_0(n,r) + kf_1(n,r)$ denote the total number of sequences satisfying the stated conditions. Then

$$S(n+1,r) = k\sum_{s=0}^{r} \binom{n-s}{s}\binom{n-2s}{r-s}(k-1)^{n-r-s} + \sum_{s=0}^{r-1} \binom{n-s}{s}\binom{n-2s}{r-s-1}(k-1)^{n-r-s+1}$$

$$+\sum_{s=0}^{r-2} \binom{n-s-1}{s}\binom{n-2s-1}{r-s-2}(k-1)^{n-r-s+1}.$$

We remark that if we sum over r we get

$$\sum_{r=0}^{n+1} S(n+1,r) = k\sum_{2s\leqslant n} \binom{n-s}{s}k^{n-2s} + \sum_{2s\leqslant n} \binom{n-s}{s}k^{n-2s} + \sum_{2s<n} \binom{n-s-1}{s}k^{n-2s-1}$$

$$= \sum_{2s\leqslant n+1} \binom{n-s+1}{s}k^{n-2s+1} + \sum_{2s\leqslant n} \binom{n-s}{s}k^{n-2s}.$$

Editorial Note: G. Wulczyn solved H-221 (previous issue).

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