ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems to Professor A.P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-310 Proposed by Daniel Finkel, Brooklyn, New York.

Find some positive integers n and r such that the binomial coefficient $\binom{n}{r}$ is divisible by $n + 1$.

B-311 Proposed by Jeffrey Shallit, Wynnewood, Pennsylvania.

Let k be a constant and let $\{a_n\}$ be defined by

$$a_n = a_{n-1} + a_{n-2} + k, \quad a_0 = 0, \quad a_1 = 1.$$

Find

$$\lim_{n \to \infty} (a_n / F_n).$$

B-312 Proposed by J.A.H. Hunter, Fun with Figures, Toronto, Ontario, Canada.

Solve the doubly-true alphametic

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  ONE
  ONE
  ONE
  TWO
THREE
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EIGHT
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Unity is not normally considered so, but here our ONE is prime!

B-313 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Let

$$M(x) = L_1 x + (L_2/2)x^2 + (L_3/3)x^3 + \cdots.$$

Show that the Maclaurin series expansion for $e^{M(x)}$ is $F_1 + F_2 x + F_3 x^2 + \cdots$.

B-314 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $L_{2p^k} \equiv 3 \pmod{10}$ for all primes $p \geq 5$.

SOLUTIONS

DIFFERENTIATING FIBONACCI GENERATING FUNCTION

B-279 (Correction of typographical error in Vol. 12, No. 1 (February 1974).

Find a closed form for the coefficient of x^n in the Maclaurin series expansion of

$$(x+2x^2)/(1-x-x^2)^2.$$

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Let

$$F(x) = (1-x-x^2)^{-1} = \sum_{n=0}^{\infty} F_{n+1}x^n$$

be the well-known generating function for the Fibonacci numbers. Differentiating term by term, we have formally:

$$F'(x) = (1+2x)(1-x-x^2)^{-2} = \sum_{n=1}^{\infty} nF_{n+1}x^{n-1} .$$

Therefore,

$$(x+2x^2)(1-x-x^2)^{-2} = \sum_{n=0}^{\infty} nF_{n+1}x^n .$$

Hence, the required coefficient is equal to nF_{n+1}, $n = 0, 1, 2, \cdots$.

Also solved by Clyde A. Bridger, Charles Chouteau, Edwin T. Hoefer, A.C. Shannon, and the Proposer.

GOLDEN POWERS OF 2

B-286 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let g be the "golden ratio" defined by

$$g = \lim_{n\to\infty} (F_n/F_{n+1}).$$

Simplify

$$\sum_{0}^{n} \binom{n}{i} g^{2n-3i} .$$

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

As $1/g = \alpha = (1+\sqrt{5})/2$ then the sum equals

$$g^{2n}\cdot\sum_{0}^{n} \binom{n}{i}\cdot(\alpha^3)^i ,$$

that is $g^{2n}\cdot(1+\alpha^3)^n$, which simplifies to 2^n.

Also solved by W.G. Brady, Paul S. Bruckman, Ralph Garfield, Frank Higgins, A.C. Shannon, Martin C. Weiss, David Zeitlin, and the Proposer.

SIMPLIFIED

B-287 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let g be as in B-286. Simplify

$$g^2\left\{(-1)^{n-1}[F_{n-3}-gF_{n-2}]+g+2\right\}.$$

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Since $g = 1/\alpha = -\beta$,

$$F_{n-3} - gF_{n-2} = 5^{-1/2}\left\{\alpha^{n-3} - \beta^{n-3} - \alpha^{-1}\alpha^{n-2} - \beta\cdot\beta^{n-2}\right\} = 5^{-1/2}\left\{\beta^{n-2}\right\}\left\{\alpha - \beta\right\} = \beta^{n-2}$$
$$= (-1)^{n-2}g^{n-2}.$$

Also, since $\beta^2 = \beta + 1$, then $g^2 = 1 - g$. Hence,

$$g^2(g+2) = (1-g)(2+g) = 2 - g - g^2 = 2 - g - 1 + g = 1.$$

Therefore, the given expression reduces to:

$$g^2(-1)^{n-1}(-1)^{n-2}g^{n-2} + 1 = 1 - g^n.$$

Also solved by Ralph Garfield, Frank Higgins, and the Proposer.

A MULTIPLE OF L_{2n}

B-288 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove that $F_{2n(4k+1)} \equiv F_{2n} \pmod{L_{2n}}$ for all integers n and k.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

If p is even then

$$F_{m+p} - F_{m-p} = L_mF_p.$$

Replace p by $4nk$ and m by $2n(2k+1)$ to get

$$F_{2n(4k+1)} = F_{2n} + L_{2n(2k+1)}F_{4nk}.$$

The required congruence follows with an application of Carlitz' result: L_a divides L_b iff $b = a(2c-1)$, $a > 1$. ("A Note on Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 2, No. 1, 1964, pp. 15–28.)

Also solved by Clyde A. Bridger, Ralph Garfield, Frank Higgins, A.C. Shannon, Gregory Wulczyn, David Zeitlin, and the Proposer.

A MULTIPLE OF L_{2n+1}

B-289 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove that $F_{(2n+1)(4k+1)} \equiv F_{2n+1} \pmod{L_{2n+1}}$, for all integers n and k.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

If p is even then

$$F_{m+p} - F_{m-p} = L_mF_p.$$

Replace p by $2k(2n+1)$ and m by $(2n+1)(2k+1)$ to get

$$F_{(2n+1)(4k+1)} - F_{2n+1} = L_{(2n+1)(2k+1)}F_{2k(2n+1)}.$$

The required congruence follows with an application of Carlitz' result: L_a divides L_b iff $b = a(2c-1)$, $a > 1$. ("A Note on Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 12, No. 1, 1964, pp. 15–28.)

Also solved by Clyde A. Bridger, Ralph Garfield, Frank Higgins, A.C. Shannon, Gregory Wulczyn, David Zeitlin, and the Proposer.

CONVOLUTED F_{2n}

B-290 Proposed by V.E. Hoggatt, Jr., San Jose State University, San Jose, California.

Obtain a closed form for

$$2n + 1 + \sum_{k=1}^{n} (2n + 1 - 2k)F_{2k}.$$

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

The sum of the first k odd indexed Fibonacci numbers is F_{2k} and that of the first k even indexed ones is $F_{2k+1} - 1$, where $k \geq 1$.

Therefore,

$$2n+1+\sum_{k=1}^{n}(2n+1-2k)F_{2k} = 2n+1+F_{2n+1}-1+2\sum_{k=1}^{n-1}(F_2+F_4+\cdots+F_{2k})$$

$$= 2n+F_{2n+1}+2\sum_{k=1}^{n-1}(F_{2k+1}-1)$$

$$= 2n+F_{2n+1}+2(F_{2n}-F_1-n+1)$$

$$= F_{2n+1}+2F_{2n} = L_{2n+1}.$$

Also solved by W.G. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Frank Higgins, A.C. Shannon, Gregory Wulczyn, and the Proposer.

TRANSLATED RECURSION

B-192 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the second-order recursion relation for $\{z_n\}$ given that

$$z_n = \sum_{k=0}^{n}\binom{n}{k}y_k \qquad \text{and} \qquad y_{n+2} = ay_{n+1}+by_n,$$

where a and b are constants.

Solution by A.C. Shannon, New South Wales Institute of Technology, N.S.W., Australia.

Let $y_n = A\alpha^n + B\beta^n$, where A,B depend on y_1, y_2 and α,β are the roots of the auxiliary equation

$$0 = x^2 - ax - b.$$

Then

$$z_n = \sum_{k=0}^{n}\binom{n}{k}(A\alpha^k+B\beta^k) = A(1+\alpha)^n+B(1+\beta)^n$$

$$= ((1+\alpha)+(1+\beta))z_{n-1}-(1+\alpha)(1+\beta)z_{n-2} = (a+2)z_{n-1}-(a-b+1)z_{n-2}$$

since $a = \alpha+\beta$ and $b = -\alpha\beta$.

Also solved by W.G. Brady, Paul S. Bruckman, Ralph Garfield, Frank Higgins, David Zeitlin, and the Proposer.

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