SPECIAL PARTITIONS

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In this paper, we discuss the partitions $p(n)$ of non-negative integers n, using summands 1 and 2. These are collections of 1 and 2 whose sum is n without regard to order.

Example.

$$5 = 2+2+1 = 1+1+1+1+1 = 2+1+1+1\ ;$$

thus $p(5) = 3$.

Theorem.

$$p(2n+1) = p(2n) = n+1 \quad \text{for} \quad n \geq 0.$$

Proof. Clearly, $p(0) = 1$, using no ones or twos, and $p(1) = 1$. First, $2n$ is the sum of n two's. Each two can be replaced by a pair of ones. This can be done in n distinct ways, making $(n+1)$ possible partitions. Second, $2n+1$ is the sum of n two's and a one. Thus, it also has $(n+1)$ distinct partitions.

Theorem. If all the partitions of n are displayed simultaneously, then there are $U(n)$ ones, $S(n)$ twos, and $P(n)$ plus signs, where

$$U(2n) = 2T_n\ ,$$

$$U(2n+1) = 2T_n + p(n)\ ,$$

$$S(2n+1) = S(2n) = T_n\,,$$

$$P(2n+1) = 3T_n\,,$$

$$P(2n+2) = 3T_n + 2n + 1\,,$$

where T_n is the n^{th} triangular number, $n \geq 0$.

Proofs. Let us start with $S(2n)$, $n \geq 0$. Clearly, there are n twos and each two is sequentially replaced by a pair of ones in succeeding partitions until there are no twos. Thus

$$S(2n) = n + (n-1) + \cdots + 2 + 1 = T_n\,.$$

Clearly, $(2n+1)$ also has n twos and a one so that the number of twos in all specialized partitions of $(2n+1)$ is also T_n.

Next, consider $N = 2n$. From the sequential construction of the partitions beginning with n twos it is clear that the number of ones is $2T_n$. However, for $N = 2n+1$ we need an extra one for each partition; thus

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