DISTRIBUTION OF THE ZEROES OF ONE CLASS OF POLYNOMIALS

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INTRODUCTION

In the present paper we shall prove that the zeroes of the real polynomials

(1) $$f_0(x) = 0, \quad f_1(x) = s, \quad f_i(x) = x, \quad f_{n+1}(x) = xf_n(x) + f_{n-1}(x), \quad n = 2, 3, \cdots$$

with $s \neq 0$ and $n \geqslant 2$ are simple, of the form $-2i\cos\theta$, where $i^2 = -1$. and if $2i\cos\theta_j^{(n+1)}, j = 1, \cdots, n$ are the zeroes of $f_{n+1}(x)$, then the points $\cos\theta_j^{(n+1)}, j = 1, \cdots, n$ are divided by $\cos\theta_j^{(n)}, j = 1, \cdots, n-1$ and for every interval between two successive points $-[\cos\theta_j^{(n+1)}, \cos\theta_{j+1}^{(n+1)}]$ one and only one of the following three possibilities holds:

(a) The interval contains one of $\cos\theta_j^{(n-k+i)}$, $1 \leqslant k \leqslant n-1, j = 1, \cdots, n-k$.

(b) It contains one of $\cos(j\pi/k)$, $j = 1, \cdots, k-1$ or

(c) One of the boundary points of it coincides with one of $\cos\theta_j^{(n-k+1)}$, and $\cos(j\pi/k)$ simultaneously.

When $s = 0$, then $f_n(x)$ becomes

$$f_0(x) = 0, \qquad f_n(x) = xu_{n-1}(x), \qquad n = 1, \cdots,$$

where $u_n(x)$ are derived from (1) for $s = 1$. $u_n(x)$ are Fibonacci polynomials.

1. ON THE ZEROES OF FIBONACCI POLYNOMIALS

From the well known formula:

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} 2^{n-2k}z^k = ((1+\sqrt{z+1})^{n+1} - (1-\sqrt{z+1})^{n+1})/2\sqrt{z+1}$$

and [2] it follows that:

(2) $$u_n(x) = (2^n\sqrt{x^2+4})^{-1}((x+\sqrt{x^2+4})^n - (x-\sqrt{x^2+4})^n), \quad n = 0, 1, 2, \cdots.$$

Then for $x = 2i\cos\theta$ we get:

(3) $$u_n(2i\cos\theta) = -(i^{n+1}\sin n\theta)/\sin\theta\,.$$

So, the numbers $2i\cos(j\pi/n)$, where j is an integer and $\sin(j\pi/n) \neq 0$, are zeroes of $u_n(x)$, $n \geqslant 2$. But only $n-1$ of them are distinct. Indeed, if j gets values j_1 and j_i and $j_1 - j_2$ is a multiple of $2n$ then

$$\cos(j_1\pi/n) = \cos(j_2\pi/n).$$

Otherwise

$$\cos((n+j)\pi/n) = \cos((n-j)\pi/n) \quad \text{for} \quad 0 \leqslant j \leqslant n.$$

Therefore the numbers $2i\cos(j\pi/n)$, $j = 1, \cdots, n-1$ are $n-1$ different zeroes of (2). Since $u_n(x)$ is a polynomial of the $n-1^{th}$ degree they are all its zeroes.

2. DISTRIBUTION OF THE ZEROES OF $f_n(x)$, $n = 2, \cdots$, WHEN $s \neq 0$

By induction it may be proved that:

$$f_n(x) = u_n(x) + (s-1)u_{n-2}(x), \quad n \geqslant 2. \tag{4}$$

Owing to (3) and (4) we have:

$$f_n(2i\cos\theta) = i^{n-1}((\sin n\theta/\sin\theta) - (s-1)(\sin(n-2)\theta)/\sin\theta).$$

Functions

$$Q_n(\cos\theta) = \sin n\theta/\sin\theta, \quad n = 1, \cdots,$$

are Tchebishev's polynomials of second class. Let

$$Q_{-2}(\cos\theta) = -1, \quad Q_0(\cos\theta) = 0 \quad \text{and} \quad P_n(\cos\theta) = Q_n(\cos\theta) - (s-1)Q_{n-2}(\cos\theta), \quad n = 1, \cdots.$$

Then the following conditions are fulfilled:

$$P_0(\cos\theta) = s, \qquad P_2(\cos\theta) = 2\cos\theta,$$

$$P_{n+1}(\cos\theta) = 2\cos\theta P_n(\cos\theta) - P_{n-1}(\cos\theta), \qquad n = 1, 2, \cdots$$

and the polynomials

$$P_0(\cos\theta), \quad P_1(\cos\theta), \cdots, P_{n+1}(\cos\theta)$$

form a Sturm's row. From [1]—the zeroes of $P_{n+1}(\cos\theta)$ are real, distinct and the zeroes of $P_n(\cos\theta)$ divide those of $P_{n+1}(\cos\theta)$. So, $f_{n+1}(x)$ has n distinct zeroes—

$$2i\cos\theta_j^{(n+1)}, \qquad j = 1, 2, \cdots, \cdots, n$$

too and the points $\cos\theta_j^{(n+2)}$, $j = 1, \cdots, n$ are divided by $\cos\theta_j^{(n)}$, $j = 1, \cdots, n-1$.

The position of the zeroes of $P_{n-k}(\cos\theta)$ in relation to those of $P_n(\cos\theta)$ can be examined by the help of the lemmas:

Lemma 1.

$$P_n(\cos\theta) = Q_k(\cos\theta)P_{n-k}(\cos\theta) - Q_{k-1}(\cos\theta)P_{n-k+1}(\cos\theta), \tag{4}$$

where n and k are positive integers and $n \geqslant 2$, $1 \leqslant k < n$.

This is proved by induction over n. It can be directly verified that it is valid for $n = 2$, $k = 1$ and for $n = 3$, $k = 1, 2$. If we assume that (4) is true for some $n - 1 > 3$, $k = 1, 2, \cdots, n-2$ and $n, k = 1, \cdots, n-1$, then

$$\begin{aligned} P_{n+1}(\cos\theta) &= 2\cos\theta P_n(\cos\theta) - P_{n-1}(\cos\theta) = 2\cos\theta(Q_k(\cos\theta)P_{n-k}(\cos\theta) - Q_{k-1}(\cos\theta)P_{n-k-1}(\cos\theta)) \\ &= Q_k(\cos\theta)P_{n-k-1}(\cos\theta) + Q_{k-1}(\cos\theta)P_{n-k-2}(\cos\theta) \\ &= Q_k(\cos\theta)P_{n-k+1}(\cos\theta) - Q_{k-1}(\cos\theta)P_{n-k}(\cos\theta) = Q_k(\cos\theta)P_{n-k+1}(\cos\theta), \end{aligned}$$

which is true for $k = 1, \cdots, n-2$. When $k = n-1$ and $k = n$, we have

$$P_{n+1}(\cos\theta) = 2\cos\theta Q_n(\cos\theta) - sQ_{n-1}(\cos\theta)$$

the validity of which is easily proved by induction over n.

Lemma 2.

$$P_{n-k}(\cos\theta_j^{(n+1)}) = Q_{k-1}(\cos\theta_j^{(n+1)})P_{n-1}(\cos\theta_j^{(n+1)}), \quad j = 1, 2, \cdots, n.$$

This can be proved by induction over k.

Owing to Lemma 1 and the results received above, the common zeroes of $P_n(\cos\theta)$ and $P_{n-k}(\cos\theta)$ are zeroes of $Q_{k-1}(\cos\theta)$. Moreover $P_n(\cos\theta)$ and $Q_{k-1}(\cos\theta)$ have no other common zeroes.

Let

$$(\cos\theta_j^{(n+1)},\ \cos\theta_{j+1}^{(n+1)}),\qquad 1 \leqslant j \leqslant n-1$$

be an interval between two successive zeroes of $P_n(\cos\theta)$ which doesn't contain any zeroes of $Q_{k-1}(\cos\theta)$.

Then

$$Q_{k-1}(\cos\theta_j^{(n+1)}),\ Q_{k-1}(\cos\theta_{j+1}^{(n+1)}) > 0$$

$$P_{n-1}(\cos\theta_j^{(n+1)}),P_{n-1}(\cos\theta_{j+1}^{(n+1)}) < 0$$

and by Lemma 2, we conclude that:

$$P_{n-k}(\cos\theta_j^{(n+1)}),\quad P_{n-k}(\cos\theta_{j+1}^{(n+1)}) < 0\ .$$

This shows that $P_{n-k}(\cos\theta)$ has an odd number of zeroes in

$$[\cos\theta_j^{(n+1)},\ \cos\theta_{j+1}^{(n+1)}]\ .$$

If $P_{n-k}(\cos\theta)$ has more than one zero in this interval, from Lemma 1 it will follow that $P_n(\cos\theta)$ has a zero in

$$(\cos\theta_j^{(n+1)},\ \cos\theta_{j+1}^{(n+1)})\ ,$$

which contradicts our assumption. Therefore every interval

$$[\cos\theta_j^{(n+1)},\ \cos\theta_{j+1}^{(n+1)}]$$

which doesn't contain a zero of $Q_{k-1}(\cos\theta)$, contains only one zero of $P_{n-k}(\cos\theta)$. In a similar way it is proved that if in

$$[\cos\theta_j^{(n+1)},\ \cos\theta_{j+1}^{(n+1)}]$$

there is no zero of $P_{n-k}(\cos\theta)$, it contains one zero of $Q_{k-1}(\cos\theta)$.

Thus we proved that in every interval between two successive points of

$$\cos\theta_j^{(n+1)},\qquad j = 1,\cdots,n$$

there is either one and only one of

$$\cos\theta_j^{(n-k+1)},\qquad j = 1,\cdots,n-k,$$

or one and only one of

$$\cos(j\Pi/k),\qquad j = 1,\cdots,k-1$$

or one of the boundary points of this interval coincides with one of

$$\cos\theta_j^{(n-k+1)},\quad j = 1,\cdots,n-k \qquad \text{and of} \qquad \cos(j\Pi/k),\quad j = 1,\cdots,k-1.$$

REFERENCES

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