THE H-CONVOLUTION TRANSFORM

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1. INTRODUCTION

We form the complete convolution array for a sequence whose generating function is

$$f(x) = \sum_{i=0}^{\infty} f_i x^i = \sum_{i=0}^{\infty} a_{i,0} x^i \tag{1.1}$$

with $f(0) = f_0 = a_{00} \neq 0$, and let

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij} x^i, \qquad j = 0, \pm 1, \pm 2, \pm 3, \cdots; \tag{1.2}$$

note that

$$a_{i,-1} = \delta_{i,0} = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0 \end{cases}.$$

This convolution array is the source of an infinite number of sequences which are intimately related to the coefficients of $f(x)$

Form a new sequence whose generating function $S_1(x)$ is given by

$$Hf(x) = S_1(x) \tag{1.3}$$

and

$$S_1(x) = \sum_{i=0}^{\infty} \frac{a_{ii}}{i+1} x^i = \sum_{i=0}^{\infty} a_i x^i . \tag{1.4}$$

We call the sequence $\{s_i\}_{i=0}^{\infty}$ the H-convolution transform of the sequence $\{f_i\}_{i=0}^{\infty}$, but it is easier to express this relationship between the generating functions. That is, $H\{f_i\}_{i=0}^{\infty} = \{s_i\}_{i=0}^{\infty}$ is expressed $Hf(x) = S_1(x)$.

In the next section we shall prove that, if $Hf(x) = S_1(x)$, then $f(xS_1(x)) = S_1(x)$ with $f(0) = S_1(0) \neq 0$. It is well known that

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{1.5}$$

defines the Catalan numbers, whose generating function is $C(x) = [1 - \sqrt{1-4x}]/2x$. Let $f(x) = 1/(1-x)$. The Catalan generating function satisfies $1 + xC^2(x) = C(x)$. This implies that $1/[1 - xC(x)] = C(x)$. That is, if

$$f(x) = 1/(1-x),$$

then

$$f(xC(x)) = 1/[1 - xC(x)] = C(x),$$

so that from Pascal's triangle generator we get the Catalan number generator; $H(1/(1-x)) = C(x)$, $C(0) = 1$.

2. LAGRANGE'S THEOREM

Lagrange's Theorem: (As in Polya and Szegö [1])

Let $f(z)$ and $\varphi(z)$ be regular about $z = 0$ and $f(0) \neq 0$, $\varphi(0) \neq 0$, and $z = \omega\varphi(z)$. Then

$$\frac{f(z)}{1-\omega\varphi'(z)} = \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \left.\frac{d^n(f(x)\varphi^n(x))}{dx^n}\right|_{x=0} .$$

Since

$$\varphi(0) \neq 0, \qquad \text{and} \qquad \omega = z/\varphi(z) = g(z),$$

if $f(z) = 1$, then we are dealing only with reversal of power series [2].

We now use Lagrange's theorem to prove our major result.

Theorem 1. Let $f(x)$ be analytic about $x = 0$, with $f(0) \neq 0$, and

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij}x^i ,$$

and let

$$S_1(x) = \sum_{i=0}^{\infty} \frac{a_{ii}}{i+1} x^i ;$$

then $f(xS_1(x)) = S_1(x)$, and $S_1(0) = f(0) \neq 0$.

Proof of Theorem 1. Let

$$xS_1(x) = \sum_{i=0}^{\infty} \frac{a_{ii}}{i+1} x^{i+1} ;$$

then

$$\frac{d}{dx}(xS_1(x)) = \sum_{i=0}^{\infty} a_{ii}x^i = \sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{d^i}{dx^i}\left.(f^{i+1}(x))\right|_{x=0}$$

which can be visualized for Lagrange's theorem as

$$\frac{d}{dx}(xS_1(x)) = \sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{d^i}{dx^i}\left.(f(x)f^i(x))\right|_{x=0},$$

or

$$\frac{d}{dx}(xS_1(x)) = \frac{f(z)}{1-xf'(z)} ,$$

with $\omega = x$ and $\varphi(z) = f(z)$. From $z = xf(z)$, $x \neq 0$,

(2.1)
$$\frac{df}{dz} = f'(z) = \frac{x - z\dfrac{dx}{dz}}{x^2} ,$$

and so

(2.2)
$$1 - xf'(z) = 1 - \frac{x - z\dfrac{dx}{dz}}{x} = \frac{z}{x}\frac{dx}{dz} .$$

Thus,

$$\frac{d}{dx}(xS_1(x)) = \frac{dz}{dx} ,$$

which implies that $xS_1(x) = z + c$; but $xS_1(x) \to 0$ and $z \to 0$ as $x \to 0$. Thus, $c = 0$ and $xS_1(x) = z$. Thus,

$$xS_1(x) = xf(xS_1(x))$$

or

$$S_1(x) = f(xS_1(x)) = f(z).$$

From $z = xf(z)$ to $z = xS_1(x)$ is a reversal of power series, and a necessary and sufficient condition for $S_1(x)$ to be regular about $x = 0$ is that $x = z/[f(z)] = g(z)$ be such that $g'(0) \neq 0$. Clearly, this is guaranteed by $f(0) \neq 0$, since

$$g'(z) = [f(z) - zf'(z)]/f^2(z) \qquad \text{and} \qquad g'(0) = 1/f(0) \neq 0.$$

See Copson [2].

We thus see that if $f(x)$ is regular about $x = 0$ and $f(0) \neq 0$, then $Hf(x) = S_1(x)$ is a function such that $f(0) = S_1(0) \neq 0$ and $f(xS_1(x)) = S_1(x)$, and $S_1(x)$ is regular about $x = 0$.

Corollary. $\varphi(z) = S(x)$, where $\varphi(xS(x)) = S(x)$.

We now proceed to another important

Theorem 2. Let $f(x)$ be regular about $x = 0$ and $f(0) \neq 0$, and

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij}x^i, \qquad j = 0, 1, 2, \cdots,$$

and

$$G_j(x) = \sum_{i=0}^{\infty} \frac{j}{i+j} a_{i,i+j-1}x^i .$$

Then $G_j(x) = S_1^j(x)$ for $j = 1, 2, 3, \cdots$.

Proof of Theorem 2.

$$x^jG_j(x) = \sum_{i=0}^{\infty} \frac{jx^{i+j}}{(i+j)i!} \left.\frac{d^i(f^j(x)f^i(x))}{dx^i}\right|_{x=0},$$

or

$$\frac{d}{dx}(x^jG_j(x)) = jx^{j-1}\sum_{i=0}^{\infty} \frac{x^i}{i!} \left.\frac{d^i(f^j(x)f^i(x))}{dx^i}\right|_{x=0}$$

$$= \frac{jx^{j-1}f^j(z)}{1-xf'(z)} = jx^{j-1}f^{j-1}(z)\frac{dz}{dx},$$

with $\omega = x$ and $f(z)$ replaced by $f^j(z)$; the last step follows from (2.2), the result in the proof of Theorem 1. Thus,

$$\frac{d}{dx}(x^jG_j(x)) = jz^{j-1}\frac{dz}{dx},$$

which implies that $x^jG_j(x) = z^j + c$. Since $x^jG_j(x)$ and $z^j \to 0$ as $x \to 0$, then $c = 0$, so that

$$G_j(x) = z^j/x^j = f^j(z) = S_1^j(x),$$

since the same hypotheses of Theorem 1 are used in Theorem 2, and there $f(z) = S_1(x)$. Thus,

$$S_1^j(x) = \sum_{i=0}^{\infty} \frac{j}{i+j}\frac{x^i}{i!} \left.\frac{d^i(f^j(x)f^i(x))}{dx^i}\right|_{x=0}$$

$$= \sum_{i=0}^{\infty} \frac{j}{i+j} a_{i,i+j-1}x^i, \qquad j = 1, 2, 3, \cdots.$$

The next theorem is harder to prove.

Theorem 3. Let $f(x)$ be regular about $x = 0$, $f(0) \neq 0$, and

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij}x^i, \qquad j = 0, 1, 2, \cdots.$$

Let

$$G_{-j}(x) = \sum_{i=0}^{\infty}{}' \frac{-j}{i-j}\frac{x^i}{i!}\left.\frac{d^i(f^{-j}(x)f^i(x))}{dx^i}\right|_{x=0},$$

where the prime indicates $i \neq j$. Then

$$G_{-j}(x) + \frac{x^j}{j!}\frac{d^j}{dx^j}\left.(S_1^{-j}(x))\right|_{x=0} = S_1^{-j}(x).$$

Proof of Theorem 3. Clearly the missing term is indeterminate since

$$\left.\frac{d^j}{dx^j}(f^0(x))\right|_{x=0} = \begin{cases} 0, \text{ if } j \neq 0\,; \\ 1, \text{ if } j = 0\,; \end{cases}$$

in either case, the missing term is 0/0. Now

$$x^{-j}G_{-j}(x) = \sum_{i=0}^{\infty}{}' \frac{-j}{i-j}\frac{x^{i-j}}{i!}\left.\frac{d^i(f^{-j}(x)f^i(x))}{dx^i}\right|_{x=0}$$

so that

$$\frac{d}{dx}(x^{-j}G_{-j}(x)) = -jx^{-j-1}\sum_{i=0}^{\infty}\frac{x^i}{i!}\left.\frac{d^i(f^{-j}(x)f^i(x))}{dx^i}\right|_{x=0}.$$

Thus, by Lagrange's theorem, with $\omega = x$, $\varphi(z) = f(z)$, and $f(z)$ replaced by $(f(z))^{-j}$, and by the result (2.2) in the proof of Theorem 1,

$$\frac{d}{dx}(x^{-j}G_{-j}(x)) = -jx^{-j-1}f^{-j-1}(z)\frac{dz}{dx} = -jz^{-j-1}\frac{dz}{dx},$$

since $z = xf(z)$, so that

$$x^{-j}G_{-j}(x) = z^{-j} + c,$$

and

$$G_{-j}(x) = f^{-j}(z) + cx^j = S_1^{-j}(x) + cx^j.$$

Recall that $G_{-j}(x)$ has a zero coefficient for x^j. Thus, we can get equality if and only if

$$c = -\frac{1}{j!}\frac{d^j}{dx^j}\left.(S_1^{-j}(x))\right|_{x=0},$$

which concludes the proof of Theorem 3.

3. APPLICATIONS OF THESE THEOREMS

The three theorems we have proved now give us an explicit set of instructions on how to convert the entire convolution array generated by the powers of $f(x)$ into the entire convolution array for $S_1(x)$.

The central falling diagonal is converted into $S_1(x)$, and the diagonals parallel to this are explicitly converted into $S_1^j(x)$ for all integral j, where $f(0) = S_1(0)$ and $f(xS_1(x)) = S_1(x)$. We have in reality explicitly derived series expansions for all $S_1^j(x)$ in terms of the entries of the convolution array for $f(x)$. This is

$$S_1^j(x) = \sum_{i=0}^{\infty} \frac{j}{i+j} a_{i,i+j-1} x^i , \tag{3.1}$$

where

$$a_{i,i+j-1} = \frac{1}{i!} \left. \frac{d^i(f^j(x)f^i(x))}{dx^i} \right|_{x=0} ,$$

for all integral j, with special attention given when $i+j = 0$, as earlier discussed. This, of course, can now be repeated any number of times.

A particularly pleasing special case of sequences of convolution arrays arises upon taking $f(x) = 1/(1-x)$, giving rise to the generating functions for the columns of Pascal's triangle. This paper proves and generalizes the results found when considering Catalan and related sequences which arose from inverses of matrices containing certain columns of Pascal's triangle [3], [4], [5], [6].

4. FURTHER GENERALIZATIONS

We can, of course, apply the convolution transform H to $f(x)$ several times. $Hf(x) = S_1(x)$ means $f(xS_1(x)) = S_1(x)$, and $H^2f(x) = S_2(x)$ means that $HS_1(x) = S_2(x)$, where $S_1(xS_2(x)) = S_2(x)$. Further, we can show $f(xS_2^2(x)) = S_2(x)$ as follows:

$$f(xS_1(x)) = S_1(x);$$

replace x by $xS_2(x)$ to obtain

$$f(xS_2(x)S_1(xS_2(x))) = f(xS_2^2(x)) = S_1(xS_2(x)) = S_2(x).$$

In general, one can show that, if

$$S_k(xS_{k+1}(x)) = S_{k+1}(x),$$

then

$$H^kf(x) = S_k(x) \qquad \text{and} \qquad f(xS_k^k(x)) = S_k(x). \tag{4.1}$$

Thus, one can secure an infinite sequence of generating functions from one generating function, $f(x)$.

We can now discuss the inverse convolution transform, H^{-1}. From $f(xS_1(x)) = S_1(x)$, we look at $S_1(x/f(x))$, replace x by $xS_1(x)$, so that

$$S_1(xS_1(x)/f(xS_1(x))) = S_1(x) = f(xS_1(x));$$

thus

$$S_1(x/f(x)) = f(x).$$

$H^{-1}S_1(x) = f(x)$ means $S_1(x/f(x)) = f(x)$. If we designate $f(x) = S_0(x)$, then

$$H^1S_0(x) = S_1(x),$$

and, in general,

$$H^kS_0(x) = S_k(x) \qquad \text{and} \qquad H^{-k}S_0(x) = S_{-k}(x), \tag{4.2}$$

generating a doubly infinite sequence of generating functions from the convolution array for $f(x) = S_0(x)$.

We now derive the explicit formulas for these.

Theorem 4.

$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j} a_{i,ki+j-1} x^i, \qquad k = 0, 1, 2, \cdots.$$

Proof of Theorem 4.

We consider the elements a_{ij} of the convolution array for $f(x)$ such that $f(0) \neq 0$ and

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij}x^i ,$$

j an integer. We proceed first for j positive.

For $S_1(x)$, the elements processed are a_{ii}; for $S_2(x)$, the elements processed are $a_{i,2i}$; and for $S_k(x)$, the elements processed are $a_{i,ki}$. This is, of course, done sequentially. Consider the element $a_{i,ki+j-1}$. We now find the sequential factors to convert it into the coefficient of x^i in $S_k^j(x)$.

First, we consider the diagonals parallel to the principal falling diagonal a_{ii}; the diagonal $S_1^{(k-1)i+j}(x)$ contains $a_{i,ki+j-1}$ and was multiplied by

$$\frac{(k-1)i+j}{ki+j} .$$

In the diagonals parallel to the $a_{i,2i}$, the diagonal $S_2^{(k-2)i+j}(x)$ contains $a_{i,ki+j-1}$ and was multiplied by an additional factor of

$$\frac{(k-2)i+j}{(k-1)i+j},$$

and so on. In the diagonals parallel to $a_{i,ki}$, $S_k^j(x)$ picked up a factor of $j/(i+j)$. Thus, for the terms of $S_k^j(x)$

$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{i+j} \cdot \frac{i+j}{2i+j} \cdot \cdots \cdot \frac{(k-1)i+j}{ki+j}\, a_{i,ki+j-1}x^i$$

$$= \sum_{i=0}^{\infty} \frac{j}{ki+j}\, a_{i,ki+j-1}x^i .$$

This can also be established by induction. Look at $a_{i,(k+1)i+j-1}$. Each factor we used before has its right subscript of a_{ij} advanced by i so that

$$S_{k+1}^j(x) = \sum_{i=0}^{\infty} \frac{j}{(k+1)i+j}\, a_{i,(k+1)i+j-1}x^i .$$

This holds for $j = 1, 2, 3, \cdots$, and concludes the proof of Theorem 4, for j positive. For $j = 0$, $S_k^0(x) = 1$. For $j \leqslant 0$, there are special problems to surmount.

Theorem 5. If $f^{-1}(xS^k(x)) = S(x)$, with $S(0) = f^{-1}(0) \neq 0$, then $S(x) = S_{-k}^{-1}(x)$.

Proof. The function $f^{-1}(x)$ induces a two-sided sequence of generating functions. From $f^{-1}(xS^k(x)) = S(x)$, we imply

$$S(x/(f^{-1}(x))^k) = f^{-1}(x)$$
$$S(xf^k(x)) = f^{-1}(x)$$
$$S^{-1}(xf^k(x)) = f(x).$$

But $S_{-k}(xf^k(x)) = f(x)$, so that $S(x) = S_{-k}^{-1}(x)$.

Theorem 6. For $j > 0$, $k > 0$,

$$S_{-k}^{-j}(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j} \frac{x^i}{i!} \left. \frac{d^i(f^{-j}(x)f^{-ki}(x))}{dx^i} \right|_{x=0} .$$

Proof. Apply Theorem 4 to the function $F(x) = f^{-1}(x)$. Thus for $j > 0$ and $k > 0$

$$S_{-k}^{-j}(x) = \sum_{i=0}^{\infty} \frac{-j}{-ki-j} \frac{x^i}{i!} \left. \frac{d^i(f^{-1}(x))^j(f^{-k}(x))^i}{dx^i} \right|_{x=0}$$

This is equivalent to the theorem.

SUMMARY:

$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j}\,\frac{x^i}{i!}\,\frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\Bigg|_{x=0} \tag{4.3}$$

now holds for $j \geqslant 1,\ k \geqslant 1$, or $j \leqslant -1,\ k \leqslant -1$. The case $j = 0,\ k \neq 0$ is routine and $k = 0$ for any j is routine.

We note that in the proof sequence of Theorem 4, there are no zero factors except when $j = 0$.

Theorem 7 (The Completion of Theorem 4).

If $f(z)$ is regular about $z = 0$ and $f(0) \neq 0$, then, for $k \neq 0$,

$$\text{(i)} \qquad S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j}\,\frac{x^i}{i!}\,\frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\Bigg|_{x=0}$$

when $-j/k \neq m$, a positive integer.

The prime below indicates $i \neq m$,

$$\text{(ii)} \qquad S_k^j(x) = \sum_{i=0}^{\infty}{}' \frac{j}{ki+j}\,\frac{x^i}{i!}\,\frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\Bigg|_{x=0} + \frac{x^m}{m!}\,\frac{d^m(f^j(x)f^{ki}(x))}{dx^m}\Bigg|_{x=0}$$

when $-j/k = m$, a positive integer.

Proof of Theorem 7. Let

$$g_j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j}\,\frac{x^i}{i!}\,\frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\Bigg|_{x=0}$$

for $j \neq 0$, and $g_0(x) = 1$.

Case (i).

$$x^{j/k}g_j(x) = \sum_{i=0}^{\infty} \frac{j/k}{i+j/k}\,\frac{x^{i+j/k}}{i!}\,\frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\Bigg|_{x=0}.$$

Taking the derivative,

$$\frac{d}{dx}\left(x^{j/k}g_j(x)\right) = \left(\frac{j}{k}x^{j/k-1}\right)\sum_{i=0}^{\infty} \frac{x^i}{i!}\,\frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\Bigg|_{x=0} \tag{4.4}$$

$$= \frac{j}{k}x^{j/k-1}f^j(z)\,\frac{x}{z}\,\frac{dz}{dx}.$$

But

$$z = x\varphi(z) = xf^k(z).$$

From the corollary to Theorem 1, $f^k(z) = S(x)$, where $f^k(xS(x)) = S(x)$. To identify $S(x)$, recall that

$$f(xG^k(x)) = G(x)$$

implies that

$$G(x) = S_k(x)$$

as defined for $f(x)$; hence,

$$S(x) = G^k(x) = S_k^k(x)$$

so that $f(z) = S_k(x)$.

Returning now to (4.4),

$$\frac{d}{dx}\,(x^{j/k}g_j(x)) = \frac{j}{k}\,x^{j/k-1}f^{j-k}(z)\,\frac{dz}{dx}\,. \,.$$

From $z/x = f^k(z)$, then $(z/k)^{1/k} = f(z)$, so that

$$f^{j-k}(z) = (z/x)^{(j-k)/k} \qquad \text{and} \qquad x^{j/k-1}f^{j-k}(z) = z^{j/k-1}\,.$$

Therefore,

$$\frac{d}{dx}\,(x^{j/k}g_j(x)) = \frac{j}{k}\,z^{j/k-1}\,\frac{dz}{dx}\,,$$

so that

$$x^{j/k}g_j(x) = z^{j/k} + C$$

$$g_j(x) = \frac{z^{j/k}}{x^{j/k}} + Cx^{-j/k}\,.$$

Thus,

$$g_j(x) = f^j(z) + Cx^{-j/k} = S_k^j(x) + Cx^{-j/k}\,.$$

From the definition of

$$g_j(x) = \sum_{i=0}^{\infty} \frac{j}{ik+j}\,\frac{x^i}{i!}\,\left.\frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\right|_{x=0}\,,$$

where $-j/k \ne m$, a positive integer, we see that $g_j(x)$ has a Maclaurin power series. Further, $S_k(x)$ is regular about $x = 0$, $S_k(0) \ne 0$, and hence $S_k^j(x)$ is regular about $x = 0$ and $S_k^j(0) \ne 0$; thus $S_k^j(x)$ also has a power series expansion. Their difference is a power series so that if $-j/k \ne m$, a positive integer, then $C = 0$, and the proof of part (i) is complete. Since $S_k^0(x) = 1$, then Theorem 7, part (i), is valid for all integral j and $S_0(x) = f(x)$ does not need such a form.

Case (ii). If $-j/k = m$, a positive integer, then

$$g_j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j}\,\frac{x^i}{i!}\,\left.\frac{d^i(f^j(x)f^{ki}(x))}{dx^i}\right|_{x=0}$$

when written as above has an indeterminate term; thus, as in the form in part (ii), it should be primed. Thus, $g_j(x)$ has no term when $ki + j = 0$, so it is necessary and sufficient that in

$$g_j(x) = S_k^j(x) + Cx^{-j/k}\,,$$

$$C = -\frac{1}{m!}\,\frac{d^m}{dx^m}\,\left.(S_k^{-mk}(x))\right|_{x=0}\,.$$

This completes the proof of part (ii).

Theorem 8. When $-j/k \ne m$, m a positive integer,

$$S_k^j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j}\,a_{i,ki+j-1}x^i\,.$$

When $-j/k = m$, m a positive integer,

$$S_k^j(x) = \sum_{i=0}^{\infty}{}' \frac{j}{ki+j} a_{i,ki+j-1}x^i + \frac{x^m}{m!} \frac{d^m}{dx^m} (S_k^{-mk}(x))\Bigg|_{x=0} .$$

Theorem 8 is simply a collection of results in terms of

$$f^{j+1}(x) = \sum_{i=0}^{\infty} a_{ij}x^i .$$

Theorem 9. Let

$$f(xS_k^k(x)) = S_k(x);$$

then

$$\sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{d^m}{dx^m} (S_k^{-k}(x))^m \Bigg|_{x=0} = f^k(x) \frac{d}{dx} (xf^{-k}(x)) .$$

Proof. Let

$$f(z) = 1, \quad z = xS_k^{-k}(z);$$

then

$$\sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{d^m}{dx^m} (S_k^{-k}(x))^m \Bigg|_{x=0} = \frac{x}{z} \frac{dz}{dx} ,$$

where

$$z = xS(x) \qquad \text{and} \qquad S_k^{-k}(xS(x)) = S(x),$$

but

$$S_k(xf^{-k}(x)) = f(x),$$

so that

$$S_k^{-k}(xf^{-k}(x)) = f^{-k}(x).$$

That is, $S(x) = f^{-k}(x)$. Further,

$$x/z = S_k^{-k}(z) = S^{-1}(x) = f^k(x),$$

so that

$$\sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{d^m}{dx^m} (S_k^{-k}(x))^m \Bigg|_{x=0} = f^k(x) \frac{d}{dx} (xf^{-k}(x)) . \tag{4.5}$$

Since $f^{j+1}(x)$ is implicit in our problem, we can express Eq. (4.5) in a better form.

$$[f(x)]^{j+1} = \sum_{i=0}^{\infty} a_{ij}x^i$$

$$f^k(x) = \sum_{i=0}^{\infty} a_{i,k-1}x^i$$

$$f^{-k}(x) = \sum_{i=0}^{\infty} a_{i,-k-1}x^i$$

$$xf^{-k}(x) = \sum_{i=0}^{\infty} a_{i,-k-1}x^{i+1}$$

$$\frac{d}{dx}(xf^{-k}(x)) = \sum_{i=0}^{\infty} (i+1)a_{i,-k-1}x^i = \sum_{i=0}^{\infty} b_{i,-k-1}x^i .$$

Let

$$f^k(x)\frac{d}{dx}(xf^{-k}(x)) = \sum_{m=0}^{\infty} A_m x^m ;$$

then

$$\text{(4.6)} \qquad A_m = \sum_{t=0}^{m} a_{t,k-1}b_{m-t,-k-1} = \sum_{t=0}^{m} (m+1-t)a_{t,k-1}a_{m-t,-k-1}, \quad k \neq 0.$$

Comment: For each $S_k^j(x)$, there is one term (when $-j/k = m$, m a positive integer) that is not easily specified by the convolution array for $f(x)$. With Theorem 9, we now know how to get that missing term in terms of the convolution array coefficients for $f(x)$ as given in Eq. (4.6).

5. FURTHER GENERALIZED IDENTITIES

The following is a consequence of Theorem 8 for $-j/k \neq m$, a positive integer.

Theorem 10 (A Generalized Identity)

Let

$$G_j(x) = \sum_{i=0}^{\infty} \frac{j}{ki+j} a_{i,ki+j-1}x^i = S_k^j(x),$$

$$G_s(x) = \sum_{i=0}^{\infty} \frac{s}{ki+s} a_{i,ki+s-1}x^i = S_k^s(x);$$

then

$$G_{s+j}(x) = \sum_{i=0}^{\infty} \frac{s+j}{ki+(s+j)} a_{i,ki+s+j-1}x^i = S_k^{s+j}(x).$$

Thus, by convolution it is true that

$$\text{(5.1)} \qquad \frac{s+j}{kn+s+j} a_{n,kn+s+j-1} = \sum_{t=0}^{n} \frac{j}{kt+j} a_{t,kt+j-1} \frac{s}{k(n-t)+s} a_{n-t,k(n-t)+s-1} .$$

Corollary 3 (Abel Convolution Formula)

Let $f(x) = e^x$ and $k = 1$ in Theorem 10; then by exponential convolution,

$$\frac{s+j}{n+s+j}(n+s+j)^n = \sum_{t=0}^{n} \binom{n}{t} \frac{j}{t+j}(t+j)^t \frac{s}{n-t+s}(n-t+s)^{n-t} .$$

Corollary 2 (Generalized Abel Convolution Formula)

Use Theorem 10 with $f(x) = e^x$ and k a positive integer; then

$$\frac{s+j}{kn+j+s}[(n+1)k+s+j-1]^n$$

$$= \sum_{t=0}^{n} \binom{n}{t} \frac{j}{kt+j}[(t+1)k+j-1]^t \frac{s}{k(n-t)+s}[(n-t+1)k+s-1]^{n-t}.$$

See Raney [14], who conjectured this form.

Corollary 3 (Hagen-Rothe Identity)

Let $f(x) = (1+x)^a$, $k = 1$, in Theorem 10; then

$$\frac{s+j}{n+s+j}\binom{a(n+s+j)}{n} = \sum_{t=0}^{n} \frac{s}{t+s}\binom{a(t+s)}{t}\frac{j}{n-t+j}\binom{a(n-t+j)}{n-t} .$$

Corollary 4 (Generalized Hagen-Rothe Identity)

Let $f(x) = (1+x)^a$ and k be a positive integer in Theorem 10; then

$$\frac{s+j}{kn+s+j}\binom{a[k(n)+s+j}{n}$$

$$= \sum_{t=0}^{n} \frac{j}{kt+j}\binom{a[(k)t+j}{t}\frac{s}{k(n-t)+s}\binom{a[k(n-t)+s}{n-t}$$

6. FINAL REMARKS

I. Schur in [8] has done much in this area. Schur [8] and Carlitz [7] give derivations of Lagrange's theorem. H. W. Gould in [13] has summarized much of what has been done earlier. There is still much that can be done for specialized functions $f(z)$.

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[Continued from Page 356.]

Proof. Since Σ is a non-discrete topology on X there exists $c \in X$ with $\{c\} \notin \Sigma$. Let Δ be the topology on X generated by

$$\Sigma \cup \{\{x\} \mid x \in X \backslash \{c\}\}$$

and notice Δ is non-discrete since $\{c\} \notin \Delta$.

Consider

$$S = \cap \{A \in \Delta \mid c \in A\}.$$

Since Δ is finite if $S = \{c\}$ then $\{c\} \in \Delta$. Thus, choose $b \in S \backslash \{c\}$. Let

$$\Gamma = \{B \subset X \mid b \in B \text{ or } c \notin B\}.$$

Let $T \in \Delta$. If $c \in T$ then $S \subset T$ and so $b \in T$ which implies $T \in \Gamma$. If $c \notin T$ then $T \in \Gamma$ by definition of Γ. Hence

$$\Sigma \subset \Delta \subset \Gamma.$$

Corollary. Every non-discrete topology on a finite set with n elements is contained in a non-discrete topology with $3(2^{n-2})$ elements.

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