AN APPLICATION OF SPECTRAL THEORY TO FIBONACCI NUMBERS

C. RYAVEC
Menlo College, Menlo Park, California and Santa Clara University, Santa Clara, California 95053

It is well known that the n^{th} Fibonacci number, a_n

$$(a_0 = a_1 = 1, \quad a_n = a_{n-1} + a_{n-2}, \quad n \geq 2)$$

can be explicitly written in the form

(1) $$a_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \quad n = 0, 1, 2, \cdots,$$

where

$$\lambda_1 = \tfrac{1}{2}(1+\sqrt{5}), \quad \lambda_2 = \tfrac{1}{2}(1-\sqrt{5}).$$

The purpose of this note is to derive Binet's formula (1) from the spectral decomposition of the matrix A, where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

First, note that for $n = 2, 3, 4, \cdots$, we have

(2) $$A^n = \begin{pmatrix} a_n & a_{n-1} \\ a_{n-1} & a_{n-2} \end{pmatrix}.$$

Second, note that since A is a symmetric matrix, there is an orthogonal matrix, $P(P^TP = I)$, such that

(3) $$P^{-1}AP = D = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} \quad ;$$

i.e., D is the diagonal matrix whose diagonal elements are the eigenvalues of A. These are the zeros of the characteristic equation $\lambda^2 - \lambda - 1 = 0$, of A. A short calcualtion reveals that

$$P = \frac{1}{d}\begin{pmatrix} 1 & \lambda_1 \\ -\lambda_1 & 1 \end{pmatrix} \quad ,$$

where $d > 0$, and

$$d^2 = 1 + \lambda_1^2 = \sqrt{5}\lambda_1 \ ;$$

i.e.,

$$\frac{1}{d^2}\begin{pmatrix} 1 & -\lambda_1 \\ \lambda_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \lambda_1 \\ -\lambda_1 & 1 \end{pmatrix} = D.$$

Raising the expression in (3) to the n^{th} power yields

$$(P^{-1}AP)^n = P^{-1}A^nP = D^n,$$

and, solving for A^n, we have

$$A^n = PD^nP^{-1}, \quad n \geq 2. \tag{4}$$

Equating the values in the upper left-hand corners of the two expressions in (4) gives formula (1) when $n \geq 2$. A simple check shows that (1) holds in fact, for all $n \geq 0$.

The above method for obtaining the explicit formula (1) is quite general; it can be used to obtain explicit formulae for terms in other linear recurrences. Unfortunately, it is not directly applicable to arithmetic sequences in prime number theory. In the case of the summatory function of $\Lambda(k)$ ($\Lambda(p^r) = \log p$ for any prime p, and

$$\Lambda(k) = 0$$

otherwise),

$$\Psi(X) = \sum_{k \leq x} \Lambda(k),$$

what seems to be needed is an operator, T, whose eigenvalues, ρ_k, are the zeros of the Riemann zeta-function. Then, given $x > 1$, we would have, on the one hand

$$\text{Trace}\left(\frac{x^T}{T}\right) = \sum_k \frac{x^{\rho_k}}{\rho_k},$$

and, on the other

$$\text{Trace}\left(\frac{x^T}{T}\right) = \sum_{j=0}^{\infty} \frac{\log^j x}{j!} (\text{Trace } T)^{j-1}.$$

An arithmetic interpretation of the right side of the last formula should yield an expression close to $x - \Psi(x)$.

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