ON ALTERNATING SUBSETS OF INTEGERS

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A finite set I of natural numbers is to be *alternating* [1] provided that there is an odd member of I between any two even members and an even member of I between any two odd members; equivalently, arranging the elements of I in increasing order yields a sequence in which consecutive elements have opposite parity. In this note we compute the number $a_{n,r}$ of alternating subsets of $\{1, 2, \cdots, n\}$ with exactly r elements, $0 \leqslant r \leqslant n$.

As a matter of notation we denote an alternating r-subset of $\{1, 2, \cdots, n\}$ by $(q_1, q_2, \cdots, q_r; n)$, where we assume $q_1 < q_2 < \cdots < q_r$.

Let $E_{n,r}$ (resp. $O_{n,r}$) be the number of alternating subsets of $\{1, 2, \cdots, n\}$ with r elements and with least element even (resp. odd). It follows that

$$a_{n,r} = E_{n,r} + O_{n,r} \qquad (1 \leqslant r \leqslant n). \tag{1}$$

For reasons which will soon become evident we set $E_{n,0} = O_{n,0} = 1$; hence, $a_{n,0} = 2$ for $n > 0$. In addition, set $a_{0,0} = 1$.

Lemma. For any positive integer m,

$$E_{m+1,r} = O_{m,r}; \qquad 0 \leqslant r \leqslant m+1.$$

Proof. The case $r = 0$ is trivial. If $r = m + 1$, then

$$E_{m+1,m+1} = 0 = O_{m,m+1}.$$

For $1 \leqslant r \leqslant m$ consider the correspondence

$$(q_1, q_2, \cdots, q_r; m+1) \leftrightarrow (q_1 - 1, q_2 - 1, \cdots, q_r - 1; m).$$

If q_1 is even then it easily follows that the number of r-subsets of $\{1, 2, \cdots, m+1\}$ with least element even equals the number of r-subsets of $\{1, 2, \cdots, m\}$ with least element odd, q.e.d.

Proposition 1. For any positive integer m, and $1 \leqslant r \leqslant m+1$,

$$a_{m+1,r} = a_{m,r-1} + a_{m-1,r}. \tag{2}$$

Proof. The case $m = 1$ is obvious, so assume $m \geqslant 2$. If $r = 1$ then

$$a_{m+1,1} = m+1 \qquad \text{while} \qquad a_{m,0} = 2, \qquad a_{m-1} = m-1;$$

hence (2) holds. For $r > 1$ we divide the r-subsets of $\{1, 2, \cdots, m+1\}$ (denoted as usual by $(q_1, q_2, \cdots, q_r; m+1)$) into two groups:

(i) $q_1 = 1$. Then $(q_2, \cdots, q_r; m+1)$ is an $(r-1)$-subset of $\{1, 2, \cdots, m+1\}$ which has an even least element, so there are $E_{m+1,r-1}$ such subsets.

(ii) $q_1 \geqslant 2$. Then the correspondence given in the previous lemma shows that the number of such r-subsets is $a_{m,r}$.

We thus conclude that

$$a_{m+1,r} = E_{m+1,r-1} + a_{m,r} \tag{3}$$

whence it follows that

$$a_{m+1,r} = E_{m+1,r-1} + E_{m,r-1} + a_{m-1,r}. \tag{4}$$

Applying the Lemma, Eq. (4) becomes

(5) $$a_{m+1,r} = O_{m,r-1} + E_{m,r-1} + a_{m-1,r}.$$

Substituting (1) in (5) yields (2), q.e.d.

We remark that (2) holds for $m = 0$ if we define $a_{n,r} = 0$ if $n < 0$ or $r < 0$.

The recurrence (2) can be solved using the standard technique of generating functions [2,3]. We first define

(6) $$A_n(x) = \sum_{k=0}^{\infty} a_{n,r}x^r .$$

Notice that $A_n(x)$ is a polynomial of degree n since $a_{n,r} = 0$ for $r > n$. Using (2) we deduce that for $n \geqslant 3$,

(7) $$A_n(x) = xA_{n-1}(x) + A_{n-2}(x),$$

while (6) and the boundary conditions on $a_{n,r}$ give

$$A_0(x) = a_{0,0} = 1$$

$$A_1(x) = a_{1,0} + a_{1,1}x = 2 + x$$

$$A_2(x) = a_{2,0} + a_{2,1}x + a_{2,2}x^2 = 2 + 2x + x^2 .$$

Set

$$A(y,x) = \sum_{n=0}^{\infty} A_n(x)y^n .$$

Then the above initial values together with (7) yield

(8) $$A(y,x) = \frac{(1+y)^2}{1 - xy - y^2} .$$

We now derive an explicit representation of $A_n(x)$. To begin, expand $1/(1 - xy - y^2)$ in a formal power series:

(9) $$\frac{1}{1 - xy - y^2} = \sum_{t=0}^{\infty} y^t(x+y)^t = \sum_{t=0}^{\infty} y^t \sum_{r=0}^{t} \binom{t}{r} x^{t-r}y^r = \sum_{t=0}^{\infty} \sum_{r=0}^{t} \binom{t}{r} x^{t-r}y^{t+r} .$$

Fix any integer $n \geqslant 0$. Then the coefficient of y^n in (9) is easily seen to be

(10) $$B_n(x) = \binom{n}{0} x^n + \binom{n-1}{1} x^{n-2} + \cdots + \binom{n-[n/2]}{[n/2]} x^{n-2[n/2]} .$$

It follows that $A_n(x)$, the coefficient of y^n in $A(y,x)$, is given by

(11) $$A_n(x) = \sum_{s=0}^{[n/2]} \binom{n-s}{s} x^{n-2s} + 2 \sum_{s=0}^{[(n-1)/2]} \binom{n-1-s}{s} x^{n-1-2s} + \sum_{s=0}^{[n/2]-1} \binom{n-2-s}{s} x^{n-2-2s}$$

$$= B_n(x) + 2B_{n-1}(x) + B_{n-2}(x) .$$

We now determine $a_{n,r}$, which, we recall is the coefficient of x^r in $A_n(x)$. We have two cases.

CASE 1. Assume $r \equiv n \pmod 2$. Then we can find $s \geqslant 0$ so that $n - r = 2s$, i.e., $s = \frac{1}{2}(n - r)$. Notice that $B_{n-1}(x)$ does not contain the term x^r. If $s = 0$, then $r = n$ and

$$a_{n,n} = \binom{n}{0} = 1;$$

otherwise we can rewrite r as $r = (n-2) - 2(s-1)$ and thus both $B_n(x)$ and $B_{n-2}(x)$ contain a term in x^r; hence

$$a_{n,r} = \binom{n - \frac{1}{2}(n-r)}{\frac{1}{2}(n-r)} + \binom{(n-2) - [\frac{1}{2}(n-r) - 1]}{\frac{1}{2}(n-r) - 1} \qquad (12)$$

Simplifying (12) we have that for $r \equiv n \pmod 2$,

$$a_{n,r} = \binom{\frac{1}{2}(n+r)}{\frac{1}{2}(n-r)} + \binom{\frac{1}{2}(n+r) - 1}{\frac{1}{2}(n-r) - 1}. \qquad (13)$$

CASE 2. Assume $r \not\equiv n \pmod 2$. Then the term x^r appears only in $B_{n-1}(x)$, so we obtain (in a fashion analogous to the one above) that

$$a_{n,r} = 2\binom{n-1-\frac{1}{2}(n-r-1)}{\frac{1}{2}(n-r-1)} .$$

That is, for $r \not\equiv n \pmod 2$,

$$a_{n,r} = 2\binom{\frac{1}{2}(n+r-1)}{\frac{1}{2}(n-r-1)} . \qquad (14)$$

We summarize these results in the following:

Proposition 2. Let $a_{n,r}$ be the number of alternating r-subsets of $\{1, 2, \cdots, n\}$.

(i) If $r \equiv n \pmod 2$,

$$a_{n,r} = \binom{\frac{1}{2}(n+r)}{\frac{1}{2}(n-r)} + \binom{\frac{1}{2}(n+r) - 1}{\frac{1}{2}(n-r) - 1} .$$

(ii) If $r \not\equiv n \pmod 2$,

$$a_{n,r} = 2\binom{\frac{1}{2}(n+r-1)}{\frac{1}{2}(n-r-1)} .$$

As a result of this development we obtain an interesting relation between the numbers $a_{n,r}$ and the Fibonacci numbers [3]:

Corollary. Let f_n be the Fibonacci sequence, i.e., $f_0 = f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$. Then we have

$$f_{n+2} = \sum_{r=0}^{n} a_{n,r}. \qquad (15)$$

Proof. Recall (see [3], p. 89) that the ordinary generating function for the sequence f_n is

$$F(y) = \sum_{n=0}^{\infty} f_n y^n = \frac{1}{1-y-y^2} . \qquad (16)$$

It follows from (8) that

$$A(y,1) = (1+y)^2 F(y) = \sum_{n=0}^{\infty} (f_n + 2f_{n-1} + f_{n-2})y^n ,$$

where $f_{-1} = f_{-2} = 0$. But from (7),

$$A(y,1) = \sum_{n=0}^{\infty} A_n(1)y^n ,$$

and

$$A_n(1) = \sum_{r=0}^{n} a_{n,r},$$

whence we conclude that

$$\sum_{r=0}^{n} a_{n,r} = f_n + 2f_{n-1} + f_{n-2} . \tag{17}$$

Using the recurrence

$$f_{n+1} = f_n + f_{n-1} ,$$

the right-hand side of (17) simplifies to f_{n+2}, which is the desired result, q.e.d.

REFERENCES

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TABLE 1
Jacobi Symbols: $b = 1$

a	(a/b)	(b/a)	$(a/-b)$	$(-b/a)$
–7	1	1	–1	1
–5	1	1	–1	–1
–3	1	1	–1	1
–1	1	1	–1	–1
1	1	1	1	1
3	1	1	1	–1
5	1	1	1	1
7	1	1	1	–1

TABLE 2
Jacobi Symbols: $b = 3$

a	(a/b)	(b/a)	$(a/-b)$	$(-b/a)$
–7	–1	–1	1	–1
–5	1	–1	–1	1
–3	0	0	0	0
–1	–1	1	1	–1
1	1	1	1	1
3	0	0	0	0
5	–1	–1	–1	–1
7	1	–1	1	1

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