SOME BINOMIAL SUMS

L. CARLITZ
Duke University, Durham, North Carolina 27706

1. Put

$$A(n) = \sum_{k=0}^{n+1} (-1)^k \left\{ \binom{n}{k} - \binom{n}{k-1} \right\}^3 , \tag{1.1}$$

where it is understood that

$$\binom{n}{-1} = \binom{n}{n+1} = 0 \qquad (n \geq 0).$$

Consideration of this sum was suggested by the following problem proposed by H. W. Gould [1]. Let

$$A_p(n) = \sum_{0 \leq 2k \leq n} (-1)^k \left\{ \binom{n}{k} - \binom{n}{k-1} \right\}^p .$$

Then

$$A_2(2m+1) = (2m+1)A_1(2m+1).$$

It is noted that this result does not hold for even n.

Since

$$A(n) = \sum_{k=0}^{n+1} (-1)^{n-k+1} \left\{ \binom{n}{n-k+1} - \binom{n}{n-k} \right\}^3 = \sum_{k=0}^{n+1} (-1)^{n-k+1} \left\{ \binom{n}{k-1} - \binom{n}{k} \right\}^3 ,$$

so that

$$A(n) = (-1)^n A(n), \tag{1.2}$$

therefore

$$A(2m+1) = 0. \tag{1.3}$$

However (1.2) gives no information about $A(2m)$. By (1.1) we have

$$A(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 - 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n}{k-1} + 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2$$

$$- \sum_{k=1}^{n+1} (-1)^k \binom{n}{k-1}^3 = 2 \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 - 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n}{k-1} + 3 \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} \binom{n}{k-1}^2 .$$

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Thus if we put

$$S_0(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3, \quad S_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n}{k-1},$$

$$S_2(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}\binom{n}{k-1}^2,$$

it is clear that

(1.4) $$A(n) = 2S_0(n) - 3S_1(n) + 3S_2(n).$$

In the next place, we have

$$S_2(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}\binom{n}{k-1}^2 = \sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n}{n-k+1}\binom{n}{n-k}^2$$

$$= \sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n}{k-1}\binom{n}{k}^2,$$

so that

(1.5) $$S_2(n) = (-1)^{n+1} S_1(n)$$

and (1.4) becomes

(1.6) $$A(n) = 2S_0(n) - 3\left\{1 + (-1)^n\right\} S_1(n).$$

In particular we have

(1.7) $$\begin{cases} A(2m) = 2S_0(2m) - 6S_1(2m) \\ A(2m+1) = 2S_0(2m+1). \end{cases}$$

It is well known (see for example [2, p. 13], [3, p. 243]) that $S_0(2m+1) = 0$, while

(1.8) $$S_0(2m) = (-1)^m \frac{(3m)!}{(m!)^3}.$$

However $S_1(n)$ does not seem to be known.

2. In order to evaluate $S_1(2m)$ we proceed as follows. We have

$$S_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \left\{\binom{n+1}{k} - \binom{n}{k}\right\} = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}^2 \binom{n+1}{k} - S_0(n)$$

$$= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}\binom{n+1}{k}\left\{\binom{n+1}{k} - \binom{n}{k-1}\right\} - S_0(n)$$

so that

(2.1) $$S_1(n) = T_0(n) - T_1(n) - S_0(n),$$

where

$$T_0(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+1}{k}^2, \qquad T_1(n) = \sum_{k=0}^{n+1} (-1)^k \binom{n}{k}\binom{n+1}{k}\binom{n}{k-1},$$

Now

$$T_1(n) = \sum_{k=0}^{n+1} (-1)^{n-k+1} \binom{n}{n-k+1} \binom{n+1}{n-k+1} \binom{n}{n-k}$$

$$= (-1)^{n+1} \sum_{k=0}^{n+1} (-1)^k \binom{n}{k-1} \binom{n+1}{k} \binom{n}{k},$$

that is,

(2.2) $$T_1(n) = (-1)^{n+1} T_1(n).$$

Therefore $T_1(2m) = 0$ and (2.1) yields

(2.3) $$S_1(2m) = T_0(2m) - S_0(2m).$$

In the next place

$$T_0(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k}^2 = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{n-k} \binom{n+1}{n-k}^2$$

$$= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1}^2 = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \left\{ \binom{n+2}{k+1} - \binom{n+1}{k} \right\}$$

$$= -(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \binom{n+1}{k} + (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+2}{k+1} \left\{ \binom{n+2}{k+1} - \binom{n+1}{k+1} \right\}$$

$$= -(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \binom{n+1}{k} + (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2$$

$$- (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k+1} \left\{ \binom{n+1}{k} + \binom{n+1}{k+1} \right\}$$

$$= -2(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k} \binom{n+1}{k+1} - (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k}^2$$

$$+ (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2,$$

so that

(2.4) $$\left\{ 1 + (-1)^n \right\} T_0(n) = -2(-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+1}{k} \binom{n+1}{k+1}$$

$$+ (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+2}{k+1}^2.$$

For $n = 2m + 1$, (2.4) gives no information about $T_0(2m + 1)$; indeed each sum on the right vanishes. For $n = 2m$, however, (2.4) becomes

(2.5)
$$2T_0(2m) = -2\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{2m+1}{k}\binom{2m+1}{k+1} + \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{2m+2}{k+1}^2 .$$

It is known [3, p. 243] that

(2.6)
$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{2m+1}{k}\binom{2m+1}{k+1} = (-1)^m\frac{(3m+1)!}{m!m!(m+1)!}$$

and

(2.7)
$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{2m+2}{k+1}^2 = (-1)^m\frac{2(3m+2)!}{m!m!(m+1)!(2m+1)} .$$

Substituting from (2.6) and (2.7) in (2.5), we get

(2.8)
$$T_0(2m) = (-1)^m\frac{(3m+1)!}{(m!)^3(2m+1)} .$$

Therefore by (2.3) and (1.8)

(2.9)
$$S_1(2m) = (-1)^m\frac{(3m)!}{m!m!(m-1)!(2m+1)} .$$

Finally, by (1.6) and (2.9),

(2.10)
$$A(2m) = -2(-1)^m\frac{(3m)!(m-1)}{(m!)^3(2m+1)} .$$

This completes the evaluation of the sum $A(2m)$. Note that we have not evaluated $S_1(2m+1)$.

3. For completeness we give a simple proof of (1.8), (2.6) and (2.7). We assume Saalschütz's theorem [2, p. 9]:

(3.1)
$$\sum_{k=0}^{n}\frac{(-n)_k(a)_k(b)_k}{k!(c)_k(d)_k} = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} ,$$

where

$$(a)_k = a(a+1)\cdots(a+k-1), \quad (a)_0 = 1$$

and

(3.2)
$$c+d = -n+a+b+1.$$

We rewrite (3.1) in the following way:

(3.3)
$$\sum_{r=0}^{j}\frac{(-j)_r(a+j)_r(b+c-a+1)_r}{r!(b+1)_r(c+1)_r} = \frac{(a-b)_j(a-c)_j}{(b+1)_j(c+1)_j} ;$$

the condition (3.2) is automatically satisfied. Multiplying both sides of (3.3) by $(a)_jx^j/j!$ and summing over j, it follows that

$$\sum_{j=0}^{\infty}\frac{(a)_j(a-b)_j(a-c)_j}{j!(b+1)_j(c+1)_j}x^j = \sum_{j=0}^{\infty}\frac{(a)_j}{j!}x^j\sum_{r=0}^{\infty}\frac{(-j)_r(a+j)_r(b+c-a+1)_r}{r!(b+1)_r(c+1)_r}$$

$$= \sum_{r=0}^{\infty}(-1)^r\frac{(a)_{2r}(b+c-a+1)_r}{r!(b+1)_r(c+1)_r}x^r\sum_{j=0}^{\infty}\frac{(a+2r)_j}{j!}x^j = \sum_{r=0}^{\infty}(-1)^r\frac{(a)_{2r}(b+c-a+1)_r}{r!(b+1)_r(c+1)_r}x^r(1-x)^{-a-2r} .$$

Now take $a = -n$ and we get

$$(3.4)\qquad \sum_{j=0}^{\infty} \frac{(-n)_j(-n-b)_j(-n-c)_j}{j!(b+1)_j(c+1)_j}\, x^j = \sum_{r=0}^{\infty} (-1)^r \frac{(-n)_{2r}(b+c+n-1)_r}{r!(b+1)_r(c+1)_r}\, x^r(1-x)^{n-2r}\,.$$

For $n = 2m$ and $x = 1$, (3.4) reduces to

$$(3.5)\qquad \sum_{j=0}^{\infty} \frac{(-2m)_j(-2m-b)_j(-2m-c)_j}{j!(b+1)_j(c+1)_j} = (-1)^m \frac{(2m)!(b+c+2m+1)_m}{m!(b+1)_m(c+1)_m}\,.$$

Now let b,c be non-negative integers. Then (3.5) yields

$$(3.6)\qquad \sum_{j=0}^{2m} (-1)^m \binom{2m}{j}\binom{2m+b+c}{j+b}\binom{2m+b+c}{j+c}$$

$$= (-1)^m \frac{(2m)!(3m+b+c)!(2m+b+c)!}{m!(m+b)(m+c)!(2m+b)!(2m+c)!}\,.$$

For $b = c = 0$ we get (1.8); for $b = 0$, $c = 1$ we get (2.6); for $b = c = 1$ we get (2.7).

REFERENCES

1. E 2395, *Amer. Math. Monthly,* 80 (1973), p. 75; solution, 80 (1973), p. 1146.
2. W. N. Bailey, *Generalized Hypergeometric Series,* Cambridge, 1935.
3. L. J. Slater, *Generalized Hypergeometric Functions,* Cambridge, 1966.

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[Continued from Page 214.]

$$\frac{1}{k}\log\frac{1+\sqrt{5}}{2}$$

as $n \to \infty$. Since this limiting value is an irrational number, the sequence (u_n) is u.d. mod 1.

REMARK. Let p and q be non-negative integers. Then the sequence

$$p,\ q,\ p+q,\ p+2q,\ 2p+3q,\ \cdots$$

or (H_n), $n = 1, 2, \cdots$ with

$$H_n = qF_{n-1} + pF_{n-2}\ (n \geqslant 3),\quad H_1 = p,\quad H_2 = q$$

possesses the property shown in Theorem 1. For if $v_n = \log H_n^{1/k}$, we have

$$v_{n+1} - v_n \to \frac{1}{k}\log\frac{1+\sqrt{5}}{2}$$

as $n \to \infty$.

Theorem 2. Let p, q, p^* and q^* be non-negative integers. Let (H_n) be the sequence

$$p,\ q,\ p+q,\ p+2q,\ 2p+3q,\ \cdots$$

and (H_n^*) the sequence

$$p^*,\ q^*,\ p^*+q^*,\ p^*+2q^*,\ 2p^*+3q^*,\ \cdots.$$

[Continued on Page 276.]