FIBONACCI NUMBERS AND UPPER TRIANGULAR GROUPS

LOUIS SHAPIRO
Howard University, Washington, D.C. 20001

In this note we call attention to the curious fact that the Fibonacci numbers arise when we look at that familiar example from group theory, the $n \times n$ nonsingular upper triangular matrices. Once incidence subgroups are defined the result follows quite easily.

Let K be any field with more than two elements and let K^* denote the nonzero elements of K. We define T_n to be the group of all nonsingular $n \times n$ upper triangular matrices over K. That is $T_n = \left\{ (a_{ij}) \mid a_{ij} = 0 \text{ if } i > j,\ a_{ii} \in K^*,\ a_{ij} \in K \right\}$. The key definition is as follows.

Definition. A subgroup, H, of T_n is an *incidence subgroup* if

(a) The relations defining H can be given entirely by specifying the domain for each a_{ij}.

(b) The two possibilities for each a_{ii} are $a_{ii} = 1$ or $a_{ii} \in F^*$.

(c) The two possibilities for a_{ij} when $i < j$ are $a_{ij} = 0$ or $a_{ij} \in F$.

Since $H \subseteq T_n$ we automatically have $a_{ij} = 0$ whenever $i > j$. By way of example we have

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix} \middle|\ a,b \in K,\ c \in K^* \right\}$$

is an incidence subgroup of T_3.

$$\left\{ \begin{pmatrix} 1 & a & -a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle|\ a \in K \right\}$$

is a subgroup but not an incidence subgroup since the (1,2) and (1,3) entries are dependent.

$$\left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle|\ a,b \in K \right\}$$

is not a subgroup.

We let G' denote the commutator subgroup of G. Then it is easily shown that

$$T'_n = \left\{ (a_{ij}) \mid a_{ii} = 1,\ a_{ij} \in F \text{ if } i < j \right\}.$$

For instance

$$T'_3 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle|\ a,b,c \in F \right\}$$

which is an incidence subgroup.

Our result is the following:

Proposition 1. The number of incidence sugroups, S, of T_n such that $S' = T'_n$ is F_{n+2}, where

$$\left\{ F_n \right\}_1^\infty = \left\{ 1, 1, 2, 3, 5, 8, \cdots \right\}$$

is the sequence of Fibonacci numbers.

Proof. We must have $T_n \supseteq S \supseteq T'_n$ so that if $S = \{(a_{ij})\}$ we then have $a_{ij} = 0$ for $i > j$, $a_{ij} \in K$ for $i < j$, and for each i we must specify either $a_{ii} = 1$ or $a_{ii} \in K^*$.

Suppose we specify $1 = a_{ii} = a_{i+1,i+1}$. Note that the commutator

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & & & a_{1n} \\ 0 & a_{22} & & & & a_{2n} \\ 0 & 0 & \ddots & & & \vdots \\ & & & 1 & a_{i,i+1} & \vdots \\ & & & 0 & 1 & \vdots \\ & & & & & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & & & b_{1n} \\ 0 & b_{22} & & & & b_{2n} \\ 0 & 0 & \ddots & & & \vdots \\ & & & 1 & b_{i,i+1} & \vdots \\ & & & 0 & 1 & \vdots \\ & & & & & b_{nn} \end{pmatrix}$$

Using block multiplication and the above computation we have

$$A^{-1}B^{-1}AB = \begin{pmatrix} 1 & c_{12} & \cdots & & & c_{1n} \\ 0 & 1 & & & & c_{2n} \\ 0 & 0 & \ddots & & & \vdots \\ & & & 1 & 0 & \vdots \\ & & & 0 & 1 & \vdots \\ & & & & & 1 \end{pmatrix}$$

and such matrices will not yield all of T'_n.

Similarly

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1-a^{-1} \\ 0 & 1 \end{pmatrix}$$

and we can generate T'_2 by choosing a appropriately.

Alternatively both

$$H_1 = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in F^*,\ b \in F\right\} \quad \text{and} \quad H_2 = \left\{\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \,\middle|\, a \in F^*,\ b \in F\right\}$$

are nonabelian. If every 2×2 block,

$$\begin{pmatrix} a_{ii} & a_{i,i+1} \\ 0 & a_{i+1,i+1} \end{pmatrix},$$

along the main diagonal is either H_1, H_2 or T_2 then $a_{i,i+1} \in F$ is specified for each i. This yields $S' = T'_n$. Thus if no two consecutive entries on the main diagonal are specified as 1's then $S' = T'_n$.

To complete the proof we need the standard result (for instance see Niven [1]) that the number of sequences of n plus and minus signs with no two minus signs adjacent is F_{n+2}.

Incidence subgroups are themselves an interesting topic. The term comes from incidence algebra as used in the study of locally finite partially ordered sets. The following facts are known. If K is finite then most normal and all characteristic subgroups of T'_n are incidence subgroups (see Weir [2]). The center or commutator subgroup of any incidence subgroup is itself an incidence subgroup. The number of normal incidence subgroups of T'_n is given by the Catalan numbers.

If the number of incidence subgroups of T'_n were known it might be useful in determining the number of finite T_0 topologies. However this is an unsolved problem for n larger than nine.

REFERENCES

1. I. Niven, *Mathematics of Choice,* Random House, 1965, New York, pp. 52–53.
2. A. Weir, "Sylow p-Subgroups of the General Linear Groups Over Fields of Characteristic *p*," *Proc. A.M.S.,* 6 (1955), pp. 454–464.

****