A NOTE ON SOME ARITHMETIC FUNCTIONS CONNECTED WITH THE FIBONACCI NUMBERS

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1. INTRODUCTION AND PRELIMINARIES

The Fibonacci numbers are defined as usual by

(1.1) $$F_0 = 1, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad (n \geqslant 2)$$

and the Lucas numbers are defined by

(1.2) $$L_0 = 1, \quad L_1 = 3, \quad L_n = L_{n-1} + L_{n-2} \quad (n \geqslant 2).$$

Recall that if x is any real number, $[x]$ is defined to be the greatest integer less than or equal to x, and $\{x\} = x - [x]$ is called the fractional part of x. Thus we have $0 \leqslant \{x\} < 1$.

In [1] and [2], L. Carlitz, V. E. Hoggatt, Jr., and R. Scoville have introduced and studied the arithmetic functions a and b which are defined by

(1.3) $$a(n) = [\alpha n], \qquad b(n) = [\alpha^2 n], \quad \text{where} \quad \alpha = \tfrac{1}{2}(1+\sqrt{5}), \text{ and } n > 0.$$

The functions a and b satisfy many relations which follow from (1.3), e.g.,

(1.4) $$b(n) = a(n) + n = a^2(n) + 1 \quad (n \geqslant 1)$$

(1.5) $$ab(n) = a(n) + b(n) = ba(n) + 1 \quad (n \geqslant 1).$$

Here, and throughout this paper, juxtaposition of functions indicates composition.

The equalities (1.4) and (1.5) are given in [1], along with many other properties of a and b.

In the present paper we show that the function a has the following property: Let $j > 0$ and let n be an integer with $n > F_{2j}$. If $a(n) \not\equiv a(n - F_{2j}) (\text{mod } F_{2j+1})$ then

$$a(n) \equiv a(n + kF_{2j}) \pmod{F_{2j+1}} \quad \text{for} \quad k = 0, 1, \cdots, L_{2j} - 2.$$

In fact, we have the stronger result, that

$$a(n + kF_{2j}) = a(n) + kF_{2j+1} \quad \text{for} \quad k = 0, 1, \cdots, L_{2j} - 2.$$

In addition, if $a(n) \not\equiv a(n - F_{2j}) \pmod{F_{2j+1}}$, then

$$a(n + L_{2j}F_{2j}) = a(n) + L_{2j}F_{2j} - 1.$$

We give conditions on n for deciding whether or not $a(n) \equiv a(n + (L_{2j} - 1)F_{2j}) \pmod{F_{2j+1}}$.

Finally, we have similar results for the Fibonacci numbers of odd index, and for the Lucas numbers.

2. VALUES OF THE FUNCTION a WHICH ARE CONGRUENT MODULO A FIXED FIBONACCI NUMBER

We shall require a few facts about the Fibonacci and Lucas numbers, which may be found in V.E. Hoggatt, Jr. [4].

If $\alpha = \tfrac{1}{2}(1+\sqrt{5})$ and $\beta = \tfrac{1}{2}(1-\sqrt{5})$ (i.e., α and β are the roots of the equation $x^2 - x - 1 = 0$), then the Fibonacci numbers, defined by (1.1), are also given by

(2.1) $$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n = 0, 1, 2, \cdots)$$

and the Lucas numbers defined by (1.2) are given by

(2.2) $$L_n = \alpha^n + \beta^n \quad (n = 0, 1, 2, \cdots).$$

Using (2.1) and (2.2), it is easy to check that

(2.3) $$\alpha F_n = F_{n+1} - \beta^n \quad (n = 0, 1, \cdots)$$

and

(2.4) $$\alpha L_n = L_{n+1} - \beta^{n-1}(1+\beta^2) \quad (n = 0, 1, \cdots).$$

The main results of the present paper are given in the next four theorems.

2.5 *Theorem.* Let n be a positive integer. Write $m = [\alpha n]$ and $\epsilon = \{\alpha n\}$. Suppose that $n > F_{2j}$ $(j > 0)$ and that $a(n - F_{2j}) \neq a(n) - F_{2j+1}$. Then

(i) $$\epsilon > 1 - \beta^{2j}$$

(ii) $$a(n + kF_{2j}) = a(n) + kF_{2j+1} \quad \text{for} \quad k = 0, 1, \cdots, L_{2j} - 2$$

(iii) If $\epsilon \geqslant 1 - \beta^{2j} + \beta^{4j}$, then
$$a(n + (L_{2j} - 1)F_{2j}) = a(n) + (L_{2j} - 1)F_{2j+1}$$

(iv) If $\epsilon < 1 - \beta^{2j} + \beta^{4j}$, then
$$a(n + (L_{2j} - 1)F_{2j}) = a(n) + (L_{2j} - 1)F_{2j+1} - 1$$

(v) $$a(n + L_{2j}F_{2j}) = a(n) + L_{2j}F_{2j+1} - 1.$$

2.6 *Theorem.* Let n be a positive integer, and set $m = [\alpha n]$ and $\epsilon = \{\alpha n\}$. Suppose that $n > F_{2j+1}$ and
$$a(n - F_{2j+1}) \neq a(n) - F_{2j+2}.$$
Then

(i) $$\epsilon < |\beta|^{2j+1}$$

(ii) $$a(n + kF_{2j+1}) = a(n) + kF_{2j+2} \quad \text{for} \quad k = 0, 1, \cdots, L_{2j+1} - 1$$

(iii) If $\epsilon < \beta^{4j+2}$, then we have
$$a(n + L_{2j+1}F_{2j+1}) = a(n) + L_{2j+1}F_{2j+2}$$

(iv) If $\epsilon \geqslant \beta^{4j+2}$, then
$$a(n + L_{2j+1}F_{2j+1}) = a(n) + L_{2j+1}F_{2j+2} + 1$$

(v) $$a(n + (L_{2j+1} + 1)F_{2j+1}) = a(n) + (L_{2j+1} + 1)F_{2j+2} + 1.$$

2.7 *Theorem.* Let n be a positive integer, and set $m = [\alpha n]$ and $\epsilon = \{\alpha n\}$. Suppose that $n > L_{2j}$ $(j > 0)$ and that $a(n - L_{2j}) \neq a(n) - L_{2j+1}$. Then

(i) $$\epsilon < |\beta|^{2j-1}(1+\beta^2)$$

(ii) $$a(n + kL_{2j}) = a(n) + kL_{2j+1} \quad \text{for} \quad k = 0, 1, \cdots, F_{2j} - 1.$$

(iii) If $\epsilon < \beta^{4j}$, then
$$a(n + F_{2j}L_{2j}) = a(n) + F_{2j}L_{2j+1}$$

(iv) If $\epsilon \geqslant \beta^{4j}$, then
$$a(n + F_{2j}L_{2j}) = a(n) + F_{2j}L_{2j+1} + 1$$

(v) $$a(n + (F_{2j} + 1)L_{2j}) = a(n) + (F_{2j} + 1)L_{2j+1} + 1.$$

2.8 *Theorem.* Let n be a positive integer, with $m = [\alpha n]$ and $\epsilon = \{\alpha n\}$. Suppose that $n > L_{2j+1}$ $(j > 0)$ and that $a(n - L_{2j+1}) \neq a(n) - L_{2j+2}$. Then

(i) $$\epsilon > 1 - \beta^{2j}(1+\beta^2)$$

(ii) $$a(n + kL_{2j+1}) = a(n) + kL_{2j+2} \quad \text{for} \quad k = 0, 1, \cdots, F_{2j+1} - 2$$

(iii) If $\epsilon > 1-\beta^{2j}(1+\beta^2)+\beta^{4j+2}$, then

$$a(n+(F_{2j+1}-1)L_{2j+1}) = a(n)+(F_{2j+1}-1)L_{2j+2}$$

(iv) If $\epsilon \leqslant 1-\beta^{2j}(1+\beta^2)+\beta^{4j+2}$, then

$$a(n+(F_{2j+1}-1)L_{2j+1}) = a(n)+(F_{2j+1}-1)L_{2j+2}-1$$

(v) $$a(n+F_{2j+1}L_{2j+1}) = a(n)+F_{2j+1}L_{2j+2}-1.$$

The proofs of Theorems 2.5–2.8 are given in §3.

It is natural to ask about the values of $a(kF_m)$ and $a(kL_m)$, and in fact, we have the following theorem (which is not quite a direct corollary of the preceding results).

2.9 *Theorem.* Let $j > 0$ be any integer. Then

(a) $$a(kF_{2j}) = kF_{2j+1}-1 \quad \text{for} \quad k = 1, 2, \cdots, L_{2j}-1$$

and $$a(L_{2j}F_{2j}) = L_{2j}F_{2j+1}-2.$$

(b) $$a(kF_{2j+1}) = kF_{2j+2} \quad \text{for} \quad k = 1, 2, \cdots, L_{2j+1}$$

and $$a((L_{2j+1}+1)F_{2j+1}) = (L_{2j+1}+1)F_{2j+1}+1.$$

(c) $$a(kL_{2j}) = kL_{2j+1} \quad \text{for} \quad k = 1, 2, \cdots, F_{2j}$$

and $$a((F_{2j}+1)L_{2j}) = (F_{2j}+1)L_{2j+1}+1$$

(d) $$a(kL_{2j+1}) = kL_{2j+2}-1 \quad \text{for} \quad k = 1, 2, \cdots, F_{2j+1}-1$$

and $$a(F_{2j+1}L_{2j+1}) = F_{2j+1}L_{2j+2}-2.$$

Proof. The proofs of all four parts are very similar, and we prove only (a). By (1.3) we have

$$a(kF_{2j}) = [k\alpha F_{2j}] = [k(F_{2j+1}-\beta^{2j})],$$

where the last equality follows from (2.3). It is easy to check, using (2.2), that

$$L_{2j}\beta^{2j} = 1+\beta^{4j} > 1,$$

while

$$(L_{2j}-1)\beta^{2j} = 1+\beta^{4j}-\beta^{2j}$$

and since $|\beta| < 1$, we have $\beta^{2j} > \beta^{4j}$, so that

$$(L_{2j}-1)\beta^{2j} < 1.$$

Then for all $k = 1, 2, \cdots, L_{2j}-1$, we have $k\beta^{2j} < 1$, while $L_{2j}\beta^{2j} > 1$. This proves (a).

3. PROOFS

We prove in detail only Theorems 2.5 and 2.7. It is then obvious how to prove Theorems 2.6 and 2.8.

Proof of Theorem 2.5. From the definition (1.3) of the function a, we have $a(n) = m$, and

$$\begin{aligned} a(n-F_{2j}) &= [\alpha(n-F_{2j})] = [\alpha n-\alpha F_{2j}] \\ &= [m+\epsilon-(F_{2j+1}-\beta^{2j})] \qquad \text{(by (2.3))} \\ &= [m-F_{2j+1}+(\epsilon+\beta^{2j})]. \end{aligned}$$

Now the assumption

$$a(n-F_{2j}) \neq a(n)-F_{2j+1}$$

implies that

$$\epsilon+\beta^{2j} > 1$$

and this proves part (i).

To see (ii), suppose that $k > 0$ is any integer. Then

$$\begin{aligned} a(n+kF_{2j}) &= [\alpha n + k(\alpha F_{2j})] \\ &= [m+\epsilon+k(F_{2j+1}-\beta^{2j})] \quad \text{(by (2.3))} \\ &= [m+kF_{2j+1}+\epsilon-k\beta^{2j}]\,. \end{aligned}$$

To prove (ii), we need only show that for all k satisfying $0 \leqslant k \leqslant L_{2j}-2$, we have

$$0 < \epsilon - k\beta^{2j} < 1. \tag{3.1}$$

By (i), we have $\epsilon > 1-\beta^{2j}$. It suffices to show

$$1-\beta^{2j} \geqslant k\beta^{2j} > 0 \quad (k = 0, 1, \cdots, L_{2j}-2) \tag{3.2}$$

or equivalently,

$$(k+1)\beta^{2j} \leqslant 1 \quad (k = 0, 1, \cdots, L_{2j}-2). \tag{3.3}$$

Clearly, if we can show

$$(L_{2j}-1)\beta^{2j} \leqslant 1, \tag{3.4}$$

the inequality (3.3) will follow. By (2.2), we have

$$L_{2j}\beta^{2j} = (\alpha^{2j}+\beta^{2j})\beta^{2j} = 1+\beta^{4j} \tag{3.5}$$

and so

$$(L_{2j}-1)\beta^{2j} = 1+\beta^{4j}-\beta^{2j}.$$

Since $|\beta| < 1$, we have $\beta^{4j} < \beta^{2j}$ for all $j > 0$, and this proves (3.4).

To see (iii) and (iv), we have

$$a(n+(L_{2j}-1)F_{2j}) = [m+\epsilon+(L_{2j}-1)F_{2j+1}-(L_{2j}-1)\beta^{2j}]\,.$$

If $0 \leqslant \epsilon-(L_{2j}-1)\beta^{2j} < 1$, then we have

$$a(n+(L_{2j}-1)F_{2j}) = a(n)+(L_{2j}-1)F_{2j+1}\,.$$

But since

$$(L_{2j}-1)\beta^{2j} = 1-\beta^{2j}+\beta^{4j},$$

then

$$0 \leqslant \epsilon-(L_{2j}-1)\beta^{2j} < 1$$

is equivalent to

$$0 \leqslant \epsilon-(1-\beta^{2j}+\beta^{4j}) < 1 \tag{3.6}$$

or equivalently,

$$0 < 1-\beta^{2j}+\beta^{4j} \leqslant \epsilon < 1 \tag{3.7}$$

(since we always have $0 < \epsilon < 1$). This proves (iii).

It is clear that if (3.6) (and hence (3.7)) does not hold, then we must have

$$\epsilon-(L_{2j}-1)\beta^{2j} < 0 \tag{3.8}$$

since $0 < \epsilon < 1$ and $(L_{2j}-1)\beta^{2j} > 0$. It is evident that if (3.8) holds, then

$$a(n+(L_{2j}-1)F_{2j}) = a(n)+(L_{2j}-1)F_{2j+1}-1.$$

This proves (iv).

Finally, to see (v), we have from (3.5) that $L_{2j}\beta^{2j} = 1+\beta^{4j}$. Then

$$a(n+L_{2j}F_{2j}) = [m+\epsilon+L_{2j}(F_{2j+1}-\beta^{2j})] = [m+L_{2j}F_{2j+1}-1+\epsilon-\beta^{4j}]$$

We must show that

$$0 < \epsilon-\beta^{4j} < 1. \tag{3.9}$$

It is easy to compute that

(3.10) $$.6 < |\beta| < .7$$

so that $\beta^2 < \frac{1}{2}$ and $\beta^4 < \frac{1}{4}$. By (i) we know $\epsilon > 1 - \beta^{2j}$, and since $j > 0$, this gives $\epsilon > \frac{1}{2}$. But also, $\beta^{4j} \leqslant \beta^4 < \frac{1}{4}$, and (3.9) follows. This proves (v) and completes the proof of Theorem 2.5.

Proof of Theorem 2.7. As before, we have $a(n) = m$, and

$$a(n - L_{2j}) = [m + \epsilon - aL_{2j}] = [m + \epsilon - (L_{2j+1} - \beta^{2j-1}(1+\beta^2))]$$

(by (2.4)). Then the assumption $a(n - L_{2j}) \neq a(n) - L_{2j+1}$ is equivalent to

(3.11) $$\epsilon + \beta^{2j-1}(1+\beta^2) < 0,$$

since $\beta < 0$. Clearly (3.11) is the same as

(3.12) $$\epsilon < |\beta|^{2j-1}(1+\beta^2)$$

and this proves (i).

To see (ii), we first have, for any integer $k > 0$,

$$a(n + kL_{2j}) = [m + \epsilon + k(L_{2j+1} - \beta^{2j-1}(1+\beta^2))].$$

As in the proof of Theorem 2.5, we need to show that

(3.13) $$0 < \epsilon + (F_{2j} - 1)|\beta|^{2j-1}(1+\beta^2) < 1.$$

We first note that, since $\alpha\beta = -1$,

(3.14) $$1 + \beta^2 = 1 - \frac{\beta}{\alpha} = \frac{\alpha - \beta}{\alpha}.$$

Then we have

$$F_{2j}|\beta|^{2j-1}(1+\beta^2) = \frac{\alpha^{2j} - \beta^{2j}}{\alpha - \beta} \cdot |\beta|^{2j-1} \cdot \frac{\alpha - \beta}{\alpha} = 1 - \frac{|\beta|^{4j-1}}{\alpha} = 1 - \beta^{4j}.$$

Then, using (i), we have (since $j > 0$)

$$0 < \epsilon + (F_{2j} - 1)|\beta|^{2j-1}(1+\beta^2)$$
$$< |\beta|^{2j-1}(1+\beta^2) + (1 - \beta^{4j}) - |\beta|^{2j-1}(1+\beta^2) = 1 - \beta^{4j} < 1.$$

It follows that if $0 \leqslant k \leqslant F_{2j} - 1$, we have

(3.16) $$0 < \epsilon + k|\beta|^{2j-1}(1+\beta^2) < 1$$

and (ii) is proved.

It is clear from (3.15) that if $0 < \epsilon < \beta^{4j}$, then

(3.17) $$0 < \epsilon + F_{2j}|\beta|^{2j-1}(1+\beta^2) = \epsilon + (1 - \beta^{4j}) < 1$$

and (iii) follows. On the other hand, if $\epsilon \geqslant \beta^{4j}$, then

$$\epsilon + F_{2j}|\beta|^{2j-1}(1+\beta^2) = \epsilon + (1 - \beta^{4j}) \geqslant 1,$$

and this proves (iv).

To see (v), we have

(3.18) $$(F_{2j} + 1)|\beta|^{2j-1}(1+\beta^2) = (1 - \beta^{4j}) + |\beta|^{2j-1}(1+\beta^2) > 1$$

and it follows that (v) holds.

This completes the proof of Theorem 2.7.

In view of (1.3), it is clear that Theorems 2.5 – 2.8 all remain true if we substitute the function b for the function a wherever it appears.

REFERENCES

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