SET PARTITIONS

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1. Let Z_n denote the set $\{1, 2, \cdots, n\}$. Let $S(n,k)$ denote the number of partitions of Z_n into k non-empty subsets $B_1, \cdots, B_k$. The B_k are called *blocks* of the partition. Put

$$n_j = |B_j| \qquad (j = 1, 2, \cdots, k),$$

so that

(1.1) $$n_1 + n_2 + \cdots + n_k = n.$$

It is convenient to introduce a slightly different notation. Put

(1.2) $$n = k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n,$$

where

$$k_j \geqslant 0 \qquad (j = 1, 2, \cdots, n)$$

and

(1.3) $$k_1 + k_2 + \cdots + k_n = k.$$

We call (1.2) a number partition of the integer n; the condition (1.3) indicates that the partition is into k parts, not necessarily distinct. For brevity (1.2) is often written in the form

(1.4) $$n = 1^{k_1} 2^{k_2} \cdots n^{k_n}.$$

Corresponding to the partition (1.2) we have

(1.5) $$\frac{n!}{(1!)^{k_1}(2!)^{k_2} \cdots (n!)^{k_n}} \; \frac{1}{k_1!\,k_2! \cdots k_n!}$$

set partitions. Hence

(1.6) $$S(n,k) = \sum \frac{1}{(1!)^{k_1}(2!)^{k_2} \cdots (n!)^{k_n}} \; \frac{1}{k_1!\,k_2! \cdots k_n!},$$

where the summation is over all nonnegative $k_1, k_2, \cdots, k_n$ satisfying (1.2) and (1.13). Thus

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} S(n,k) z^k = \sum_{k_1, k_2, \cdots = 0}^{\infty} \left(\frac{x}{1!}\right)^{k_1} \left(\frac{x^2}{2!}\right)^{k_2} \cdots \frac{z^{k_1}}{k_1!} \frac{z^{k_2}}{k_2!} \cdots$$

$$= \sum_{k_1, k_2, \cdots = 0}^{\infty} \frac{1}{k_1!} \left(\frac{xz}{1!}\right)^{k_1} \frac{1}{k_2!} \left(\frac{x^2 z}{2!}\right)^{k_2} \cdots$$

$$= \exp\left(xz + \frac{x^2 z}{2!} + \frac{x^3 z}{3!} + \cdots\right)$$

and we get the well known formula

(1.7) $$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} S(n,k) \frac{x^n}{n!} z^k = \exp(z(e^x - 1)).$$

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It is clear from (1.7) that

$$\sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k, \tag{1.8}$$

which implies

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} j^n, \tag{1.9}$$

the familiar formula for a Stirling number of the second kind.

Next put

$$A_n(z) = \sum_{k=0}^{n} S(n,k)z^k \tag{1.10}$$

and in particular

$$A_n = A_n(1) = \sum_{k=0}^{n} S(n,k). \tag{1.11}$$

The polynomial $A_n(z)$ is called a single-variable Bell polynomial. The number A_n is evidently the total number of set partitions of Z_n.

From (1.7) and (1.10) we have

$$\sum_{n=0}^{\infty} A_n(z)\frac{x^n}{n!} = exp(z(e^x - 1)). \tag{1.12}$$

Differentiation with respect to x gives

$$A_{n+1}(z) = z\sum_{r=0}^{\infty}\binom{n}{r}A_r(z) \tag{1.13}$$

while differentiation with respect to z gives

$$A_n'(z) = \sum_{r=0}^{n-1}\binom{n}{r}A_r(z). \tag{1.14}$$

Hence

$$A_{n+1}(z) = zA_n(z) + zA_n'(z). \tag{1.15}$$

By (1.10), (1.15) is equivalent to the familiar recurrence

$$S(n+1, k) = S(n, k-1) + kS(n, k).$$

If we take $z = 1$ in (1.13) we get

$$A_{n+1} = \sum_{r=0}^{n}\binom{n}{r}A_r \qquad (A_0 = 1). \tag{1.16}$$

This recurrence can be proved directly in the following way. Consider a partition of Z_{n+1} into k blocks $B_1, B_2, \cdots, B_k$. Assume that the element $n+1$ is in B_k and let B_k contain r additional elements, $r \geqslant 0$. Keeping these r elements fixed it is clear that $B_1, \cdots, B_{k-1}$ furnishes a partition of Z_{n-r} into $k-1$ blocks. Since the r elements in B_k can be chosen in $\binom{n}{r}$ ways we get

$$A_{n+1} = \sum_{r=0}^{n}\binom{n}{r}A_{n-r} = \sum_{r=0}^{n}\binom{n}{r}A_r.$$

For a detailed discussion of the numbers A_n see [5]. The polynomial $A_n(z)$ is discussed in [1].

We now define (compare [4, Ch. 4])

$$S_1(n,k) = \sum \frac{n!}{1^{k_1}2^{k_2}\cdots n^{k_n}} \frac{1}{k_1!\,k_2!\cdots k_n!}, \tag{1.17}$$

where again the summation is over all nonnegative k_1, k_2, k_n satisfying (1.2) and (1.3). This definition should be compared with (1.6). It follows from (1.17) that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} S_1(n,k)z^k = \sum_{k_1,k_2,\cdots=0}^{\infty} \frac{1}{k_1!}\left(\frac{xz}{1}\right)^{k_1} \frac{1}{k_2!}\left(\frac{x^2z}{2}\right)^{k_2} \cdots$$

$$= exp\left(xz + \frac{x^2z}{2} + \frac{x^3z}{3} + \cdots\right)$$

$$= exp\left(z \log \frac{1}{1-x}\right),$$

so that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} S_1(n,k) \frac{x^n}{n!} z^k = (1-x)^{-z}. \tag{1.18}$$

It follows that

$$\sum_{k=0}^{n} S_1(n,k)z^k = z(z+1)\cdots(z+n-1), \tag{1.19}$$

and therefore $S_1(n,k)$ is a Stirling number of the first kind.

We may restate (1.17) in the following way. Let

$$B_1, B_2, \cdots, B_k \tag{1.20}$$

denote a typical partition of Z_n into k blocks with $n_j = |B|_j$. Then

$$S_1(n,k) = \qquad (n_1-1)!(n_2-1)!\cdots(n_k-1)!, \tag{1.21}$$

where the summation is over all partitions (1.20) such that

$$n_1 + n_2 + \cdots + n_k = n.$$

2. We again consider the number partition

$$n = k_1\cdot 1 + k_2\cdot 2 + \cdots + k_n\cdot n \qquad (k_1 + \cdots + k_n = k). \tag{2.1}$$

This may be replaced by

$$n = n_1 + n_2 + \cdots + n_k, \tag{2.2}$$

where

$$n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k. \tag{2.3}$$

If there are no other conditions the partition is said to be *unrestricted.* We may, on the other hand, assume that

$$n_1 > n_2 > \cdots > n_k, \tag{2.4}$$

in which case we speak of partitions into unequal parts. Alternatively we may assume that in (2.2) the parts n_j are odd. If $q(n)$ denotes the number of partitions into distinct parts and $r(n)$ the number of partitions into odd parts, it is well known that [3, Ch. 19]

$$q(n) = r(n). \tag{2.5}$$

This discussion suggests the following two problems for set partitions.

1. Determine the number of set partitions into k blocks of unequal length.
2. Determine the number of set partitions into k blocks, the number of elements in each block being odd.

We shall first discuss Problem 2. The results are similar to those of § 1 above. Let $U(n,k)$ denote the number of set partitions of Z_n into k blocks

(2.6) $$B_1, B_2, \cdots, B_k$$

with

(2.7) $$n_j = |B_j| \equiv 1 \pmod 2 \qquad (j = 1, 2, \cdots, k).$$

In addition we define $V(n,k)$ as the number of set partitions of Z_n into k blocks (2.6) with

(2.8) $$n_j = |B_j| \equiv 0 \pmod 2 \qquad (j = 1, 2, \cdots, k).$$

(In the case of number partitions, the number of partitions

$$n = n_1 + n_2 + \cdots + n_k,$$

where

$$n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k, \qquad n_j \equiv 0 \pmod 2,$$

is of course equal to the number of unrestricted partitions of $n/2$.)

Exactly as in (1.6) we have

(2.9) $$U(n,k) = \sum \frac{n!}{(1!)^{k_1}(3!)^{k_2}\cdots}\,\frac{1}{k_1!\,k_2!\cdots},$$

where the summation is over all nonnegative $k_1, k_2, \cdots$ such that

(2.10) $$\begin{cases} n = k_1\cdot 1 + k_2\cdot 3 + k_3\cdot 5 + \cdots \\ k = k_1 + k_2 + k_3 + \cdots \end{cases}$$

Similarly we have

(2.11) $$V(n,k) = \sum \frac{n!}{(2!)^{k_1}(4!)^{k_2}\cdots}\,\frac{1}{k_1!\,k_2!\cdots},$$

where now the summation is over all nonnegative $k_1, k_2, \cdots$ such that

(2.12) $$\begin{cases} n = k_1\cdot 2 + k_2\cdot 4 + k_3\cdot 6 + \cdots \\ k = k_1 + k_2 + k_3 + \cdots \end{cases}.$$

It follows from (2.9) and (2.10) that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} U(n,k)z^k = \sum_{k_1,k_2,\cdots=0}^{\infty} \frac{1}{k_1!}\left(\frac{xz}{1!}\right)^{k_1} \frac{1}{k_2!}\left(\frac{x^3z}{3!}\right)^{k_2} \cdots$$

$$= \exp\left\{z\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\right\},$$

so that

(2.13) $$\sum_{n=0}^{\infty}\sum_{k=0}^{n} U(n,k)\frac{x^n}{n!}z^k = \exp(z \sinh x).$$

The corresponding generating function for $V(n,k)$ is

(2.14) $$\sum_{n=0}^{\infty}\sum_{k=0}^{n} V(n,k)\frac{x^n}{n!}z^k = \exp(z(\cosh x - 1)).$$

It is evident from the definitions that

$$U(n,k) = 0 \quad (n \equiv k+1 \pmod 2), \qquad V(n,k) = 0 \quad (n \equiv 1 \pmod 2).$$

Corresponding to the polynomial $A_n(z)$ and the number A_n we define

$$(2.15)\qquad \begin{cases} U_n(z) = \sum_{k=0}^{n} U(n,k)z^k \\ U_n = U_n(1) = \sum_{k=0}^{n} U(n,k) \end{cases}$$

and

$$(2.16)\qquad \begin{cases} V_n(z) = \sum_{k=0}^{n} V(n,k)z^k \\ V_n = V_n(1) = \sum_{k=0}^{n} V(n,k). \end{cases}$$

Clearly U_n is the total number of set partitions satisfying (2.6) and (2.7), while V_n is the total number of set partitions satisfying (2.6) and (2.8).

By (2.13) and (2.15) we have

$$(2.17)\qquad \sum_{n=0}^{\infty} U_n(z)\frac{x^n}{n!} = exp\,(z \sinh x)$$

and by (2.14) and (2.15)

$$(2.18)\qquad \sum_{n=0}^{\infty} V_n(z)\frac{x^n}{n!} = exp\,(z\,(\cosh x - 1)).$$

Differentiating (2.17) with respect to x we get

$$\sum_{n=0}^{\infty} U_{n+1}(z)\frac{x^n}{n!} = z \cosh x\; exp\,(z \sinh x).$$

This implies

$$(2.19)\qquad U_{n+1}(z) = z \sum_{2r \leq n} \binom{n}{2r} U_{n-2r}(z).$$

Differentiation of (2.17) with respect to z gives

$$\sum_{n=0}^{\infty} U'_n(z)\frac{x^n}{n!} = \sinh x\; exp\,(z \sinh x)$$

so that

$$(2.20)\qquad U'_n(z) = \sum_{2r<n} \binom{n}{2r+1} U_{n-2r-1}(z).$$

Put $F(x,z) = exp\,(z \sinh x)$. Since

$$\frac{\partial^2}{\partial z^2}F(x,z) = \sinh^2 x F(x,z),$$

$$\frac{\partial^2}{\partial x^2}F(x,z) = \frac{\partial}{\partial x}(z \cosh x)F(x,z) = (z^2 \cosh^2 x + z \sinh x)F(x,z),$$

it follows that

$$\frac{\partial^2}{\partial x^2}F(x,z) = z^2F(x,z) + z\frac{\partial}{\partial z}F(x,z) + z^2\frac{\partial^2}{\partial z^2}F(x,z).$$

This implies

(2.21) $$U_{n+2}(z) = z^2U_n(z) + zU_n'(z) + z^2U_n''(z) = z^2U_n(z) + (zD_z)^2U_n(z)$$

and therefore

(2.22) $$U(n+2,k) = U(n,k-2) + k^2U(n,k).$$

This splits into the following pair of recurrences

(2.23) $$\begin{cases} U(2n+2,2k) = U(2n,2k-2) + 4k^2U(2n,2k) \\ U(2n+1,2k+1) = U(2n-1,2k-1) + (2k+1)^2U(2n-1,2k+1). \end{cases}$$

To get explicit formulas for $U(n,k)$ we return to (2.13). We have

$$\exp(z\sinh x) = \sum_{k=0}^{\infty}\frac{(z/2)^k}{k!}(e^x - e^{-x})^k = \sum_{k=0}^{\infty}\frac{(z/2)^k}{k!}\sum_{j=0}^{k}(-1)^k\binom{k}{j}e^{(k-2j)x}$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{n}\frac{(z/2)^k}{k!}\sum_{j=0}^{k}(-1)^k\binom{k}{j}(k-2j)^n,$$

which yields

(2.24) $$U(n,k) = \frac{1}{2^kk!}\sum_{j=0}^{k}(-1)^k\binom{k}{j}(k-2j)^n.$$

Similarly, since $\cosh x - 1 = 2\sinh^2 \tfrac{1}{2}x$,

$$\exp(z(\cosh x - 1)) = \exp(2\sinh^2 \tfrac{1}{2}x) = \sum_{k=0}^{\infty}\frac{(z/2)^k}{k!}(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x})^{2k}$$

$$= \sum_{k=0}^{\infty}\frac{(z/2)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{2k}{j}e^{(k-j)x}$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{2k\leqslant n}\frac{(z/2)^k}{k!}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}(k-j)^n,$$

we get

(2.25) $$V(n,k) = \frac{1}{2^kk!}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}(k-j)^n.$$

Comparing (2.25) with (2.24), we get

(2.26) $$V(2n,k) = \frac{(2k)!}{2^{2n-k}k!}U(2n,2k).$$

Thus the first of (2.23) gives

(2.27) $$V(2n+2,k) = (2k-1)V(2n,k-1) + k^2V(2n,k).$$

If we put

(2.28) $$V(2n,k) = \frac{(2k)!}{2^kk!}V'(n,k),$$

(2.27) becomes

(2.29) $$V'(n+1,k) = V'(n,k-1) + k^2 V'(n,k).$$

Returning to (2.18) we have

$$\sum_{n=0}^{\infty} V_{n+1}(z)\frac{x^n}{n!} = z \sinh x \exp (z (\cosh x - 1)).$$

This implies

(2.30) $$V_{n+1}(z) = z \sum_{2r<n} \binom{n}{2r+1} V_{n-2r-1}(z) .$$

Differentiation of (2.18) with respect to z gives

$$\sum_{n=0}^{\infty} V'_n(z)\frac{x^n}{n!} = (\cosh x - 1) \exp (z \cosh x - 1)$$

which implies

(2.31) $$V'_n(z) = \sum_{0<r\leqslant 2n} \binom{n}{2r} V_{n-2r}(z).$$

It is evident from (2.15) and (2.19) that

(2.32) $$U_{n+1} = \sum_{2r\leqslant n} \binom{n}{2r} U_{n-2r} .$$

Similarly from (2.30) and (2.16) we have

(2.33) $$V_{n+1} = \sum_{2r<n} \binom{n}{2r+1} V_{n-2r-1} .$$

Since $V_n = 0$ unless n is even, we may replace (2.33) by

(2.34) $$V_{2n+2} = \sum_{r=0}^{n} \binom{2n+1}{2r+1} V_{2n-2r} .$$

It is easy to prove (2.32) and (2.34) directly by a combinatorial argument, exactly like the combinatorial proof of (1.16).

The first few values of U_n, V_{2n} follow.

$$U_0 = U_1 = U_2 = 1, \quad U_3 = 2, \quad U_4 = 5, \quad U_5 = 12, \quad U_6 = 36,$$

$$V_0 = V_2 = 1, \quad V_4 = 4, \quad V_6 = 31, \quad V_8 = 379.$$

The following values of $U(2n, 2k)$, $V'(n,k)$, $V(2n+1, 2k+1)$ are computed by means of (2.23) and (2.29).

$U(2n, 2k)$

n \ k	1	2	3	4
1	1			
2	4	1		
3	16	20	1	
4	64	336	56	1

$U(2n+1, 2k+1)$

n \ k	0	1	2	3	4
0	1				
1	1	1			
2	1	10	1		
3	1	91	35	1	
4	1	820	966	84	1

$V'(n, k)$

n \ k	0	1	2	3	4
0	1				
1	1	1			
2	1	5	1		
3	1	21	14	1	
4	1	85	147	30	1

For additional properties of $U(n,k)$ see [2].

3. Put

$$P_n(z) = \sum_{k=0}^{n-1} U(2n-1, 2k+1)z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2). \tag{3.1}$$

Then, by the second of (2.23),

$$z^2P_n(z) = \sum_{k=0}^{n-1} U(2n-1, 2k+1)z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2)[z^2-(2k+1)^2-(2k+1)^2]$$

$$= \sum_{k=0}^{n} [U(2n-1, 2k-1)+(2k+1)^2U(2n-1, 2k+1)]z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2)$$

$$= \sum_{n=0}^{n} U(2n+1, 2k+1)z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2),$$

so that

$$z^2P_n(z) = P_{n+1}(z).$$

Since $P_1(z) = z$, it follows that $P_n(z) = z^{2n-1}$ and (3.1) becomes

$$z^{2n-1} = \sum_{k=0}^{n-1} U(2n-1, 2k+1)z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2). \tag{3.2}$$

Similarly it follows from the first of (2.23) that

$$z^{2n-1} = \sum_{k=0}^{n-1} U(2n, 2k)z(z^2-2^2)(z^2-4^2)\cdots(z^2-(2k-2)^2). \tag{3.3}$$

By (2.26), (2.28) and (3.2) we have also

$$(3.4)\qquad z^{2n-1} = \sum_{k=0}^{n-1} V_n'(n,k)z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2k-1)^2).$$

Formula (1.17) for $S_1(n,k)$ suggests the following definitions.

$$(3.5)\qquad U_1(n,k) = \sum \frac{n!}{1^{k_1}3^{k_2}5^{k_3}}\,\frac{1}{k_1!\,k_2!\,k_3!\cdots},$$

where the summation is over all nonnegative $k_1, k_2, k_3, \cdots$, such that

$$\begin{cases} n = k_1\cdot 1 + k_2\cdot 3 + k_3\cdot 5 + \cdots \\ k = k_1 + k_2 + k_3 + \cdots \end{cases};$$

$$(3.6)\qquad V_1(n,k) = \sum \frac{n!}{2^{k_1}4^{k_2}6^{k_3}\cdots}\,\frac{1}{k_1!\,k_2!\,k_3!},$$

where the summation is over all nonnegative $k_1, k_2, k_3, \cdots$ such that

$$\begin{cases} n = k_1\cdot 2 + k_2\cdot 4 + k_3\cdot 6 + \cdots \\ k = k_1 + k_2 + k_3 + \cdots \end{cases}.$$

We observe that $U_1(n,k)$ is the number of permutations of Z_n with k cycles each of odd length while $V_1(n,k)$ is the number of permutations of Z_n with k cycles each of even length.

It follows from (3.5) that

$$(3.7)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{n} U_1(n,k)\frac{x^n}{n!}z^k = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}z}.$$

Similarly, by (3.6),

$$(3.8)\qquad \sum_{n=0}^{\infty}\sum_{2k\le n} V_1(n,k)\frac{x^n}{n!}z^k = (1-x^2)^{-\frac{1}{2}z},$$

so that

$$(3.9)\qquad V_1(n,k) = \frac{(2n)!}{2^k n!}S_1(n,k).$$

This is also clear if we compare (3.6) with (1.17).

It is easily verified that

$$(1-x^2)\frac{\partial}{\partial x}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}z} = z\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}z}.$$

If we put

$$U_{1,n}(z) = \sum_k U_1(n,k)z^k$$

it follows from (3.7) that

$$(3.10)\qquad U_{1,n+1}(z) - n(n-1)U_{1,n-1}(z) = zU_{1,n}(z).$$

This is equivalent to

$$(3.11)\qquad U_1(n+1,k) = U_1(n,k-1) + n(n-1)U_1(n-1,k).$$

Notice that this recurrence is somewhat different in form from the familiar recurrence for $S_1(n,k)$.

By expanding the right member of (3.7) we get

(3.12) $$U_{1,n}(z) = n! \sum_{r=1}^{n} 2^r \binom{n-1}{r-1} \binom{\frac{1}{2}z}{r} \qquad (n \geq 1).$$

To verify directly that (3.12) implies (3.10) we take

$$zU_{1,n}(z) = n! \sum_{r=1}^{n} 2^r \binom{n-1}{r-1} \left\{ 2(r+1) \binom{\frac{1}{2}z}{r+1} + 2r \binom{\frac{1}{2}z}{r} \right\}$$

$$= n! \sum_{r=1}^{n} 2^r \binom{\frac{1}{2}z}{r} \left\{ 2r \binom{n-1}{r-1} + r \binom{n-1}{r-2} \right\}.$$

On the other hand

$$U_{1,n+1}(z) - n(n-1)U_{1,n-1}(z) = (n+1)! \sum_{r=1}^{n+1} 2^r \binom{n}{r-1} \binom{\frac{1}{2}z}{r} - (n-1)n! \sum_{r=1}^{n-1} 2^r \binom{n-2}{r-1} \binom{\frac{1}{2}z}{r}$$

$$= n! \sum_{r=1}^{n} 2^r \binom{\frac{1}{2}z}{r} \left\{ (n+1) \binom{n}{r-1} - (n-1) \binom{n-2}{r-1} \right\}$$

$$= n! \sum_{r=1}^{n} 2^r \binom{\frac{1}{2}z}{r} \left\{ 2r \binom{n-1}{r-1} + r \binom{n-1}{r-2} \right\}.$$

It is evident from (3.5) that

(3.13) $$U_1(n,k) = 0 \qquad (n \equiv k+1 \pmod 2).$$

This is also clear from either (3.10) or (3.11).

By means of (3.10) we get

$$U_{1,1}(z) = z, \quad U_{1,2}(z) = z^2, \quad U_{1,3}(z) = 2z + z^3,$$
$$U_{1,4}(z) = 8z^2 + z^4, \quad U_{1,5}(z) = 24z + 20z^3 + z^5.$$

The number

(3.14) $$U_{1,n} = U_{1,n}(1) = \sum_k U_1(n,k)$$

evidently denotes the total number of permutations of Z_n into cycles of odd length. By (3.12) we have

(3.15) $$U_{1,n} = n! \sum_{r=1}^{n} 2^r \binom{n-1}{r-1} \binom{\frac{1}{2}}{r} \qquad (n \geq 1).$$

Alternatively, by (3.7) and (3.17),

$$\sum_{n=0}^{\infty} U_{1,n} \frac{x^n}{n!} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = (1+x)(1-x^2)^{-\frac{1}{2}} = (1+x) \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{x}{2}\right)^{2n},$$

which yields

(3.16) $$U_{1,2n} = (2n)! \binom{2n}{n} 2^{-2n} = (1.3.5 \cdots (2n-1))^2,$$

(3.17) $$U_{1,2n+1} = (2n+1)! \binom{2n}{n} 2^{-2n} = (2n+1)U_{1,2n}.$$

4. To obtain an array orthogonal to $U(n,k)$ we consider the expansion

$$(\sqrt{1+x^2}-x)^{-z} = \sum_{n=0}^{\infty} C_n(z)\frac{x^n}{n!} \quad (4.1)$$

If we denote the left member of (4.1) by F, we have

$$\frac{\partial F}{\partial x} = \frac{z}{\sqrt{1+x^2}}F, \qquad \frac{\partial^2 F}{\partial x^2} = \left(\frac{z^2}{1+x^2} - \frac{xz}{(1+x^2)^{3/2}}\right)F,$$

which gives

$$(1+x^2)\frac{\partial^2 F}{\partial x^2} + x\frac{\partial F}{\partial x} = z^2 F. \quad (4.2)$$

Substituting from (4.1) in (4.2) we get

$$C_{n+2}(z) + n(n-1)C_n(z) + nC_n(z) = z^2 C_n(z),$$

so that

$$C_{n+2}(z) = (z^2 - n^2)C_n(z). \quad (4.3)$$

Since $C_0(z) = 1$, $C_1(z) = z$, it follows that

$$\begin{cases} C_{2n}(z) = z^2(z^2-2^2)(z^2-4^2)\cdots(z^2-(2n-2)^2) \\ C_{2n+1}(z) = z(z^2-1^2)(z^2-3^2)\cdots(z^2-(2n-1)^2). \end{cases} \quad (4.4)$$

Therefore (4.1) becomes

$$(\sqrt{1+x^2}-x)^{-z} = \sum_{n=0}^{\infty}\frac{z^2(z^2-2^2)\cdots(z^2(2n-2)^2)}{(2n)!}x^{2n} + \sum_{n=0}^{\infty}\frac{z(z^2-1^2)\cdots(z^2-(2n-1)^2)}{(2n+1)!}x^{2n+1}. \quad (4.5)$$

If we differentiate both sides of (4.5) with respect to z and then put $z = 0$, we get

$$\log(\sqrt{1+x^2}-x) = -\sum_{n=0}^{\infty}(-1)^n\frac{1^2\cdot 3^2\cdots(2n-1)^2}{(2n+1)!}x^{2n+1}.$$

Thus (4.5) becomes

$$\exp\left\{z\sum_{n=0}^{\infty}(-1)^n\frac{1^2\cdot 3^2\cdots(2n-1)^2}{(2n+1)!}x^{2n+1}\right\} \quad (4.6)$$

$$= \sum_{n=0}^{\infty}\frac{z^2(z^2-2^2)\cdots(z^2-(2n-2)^2)}{(2n)!}x^{2n} + \sum_{n=0}^{\infty}\frac{z(z^2-1^2)\cdots(z^2-(2n-1)^2)}{(2n+1)!}x^{2n+1}.$$

Now replace x by ix and z by $-iz$ and we get

$$(4.7)\quad \exp\left\{z\sum_{n=0}^{\infty} 1^2\cdot 3^2\cdots(2n-1)^2\,\frac{x^{2n+1}}{(2n+1)!}\right\} = \sum_{n=0}^{\infty}\frac{z^2(z^2+2^2)\cdots(z^2+(2n-2)^2)}{(2n)!}\,x^{2n}$$

$$+\sum_{n=0}^{\infty}\frac{z(z^2+1^2)(z^2+3^2)\cdots(z^2+(2n-1)^2)}{(2n+1)!}\,x^{2n+1}\;.$$

We now define $W(n,k)$ by means of

$$(4.8)\quad \begin{cases} z^2(z^2+2^2)(z^2+4^2)\cdots(z^2+(2n-2)^2) = \sum_{k=0}^{n} W(2n,2k)z^{2k} \\ z(z^2+1^2)(z^2+3^2)\cdots(z^2+(2n-1)^2) = \sum_{k=0}^{n} W(2n+1,2k+1)z^{2k+1}\;. \end{cases}$$

It follows at once from (3.2), (3.3) and (4.8) that

$$(4.9)\quad \sum_{j=k}^{n}(-1)^{n-j}W(2n,2j)U(2j,2k) = \sum_{j=k}^{n}(-1)^{j-k}U(2n,2j)W(2j,2k) = \delta_{n,k}\,,$$

$$(4.10)\quad \sum_{j=k}^{n}(-1)^{n-j}W(2n+1,2j+1)U(2j+1,2k+1)$$

$$= \sum_{j=k}^{n}(-1)^{j-k}U(2n+1,2j+1)W(2j+1,2k+1) = \delta_{n,k}\;.$$

By means of (4.7) we can exhibit $W(n,k)$ in a form similar to (2.9) and (2.11). Indeed it is evident from (4.7) and (4.8) that

$$(4.11)\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} W(n,k)\frac{x^n}{n!}z^k = \exp\left\{z\sum_{n=0}^{\infty} f(n)\,\frac{x^{2n+1}}{(2n+1)!}\right\}\,,$$

where for brevity we put

$$f(n) = 1^2\cdot 3^2\cdot 5^2\cdots(2n-1)^2\,.$$

It follows from (4.11) that

$$(4.12)\quad W(n,k) = \sum \frac{n!}{(1!)^{k_1}(3!)^{k_2}(5!)^{k_3}\cdots}\;\frac{(f(1))^{k_1}(f(2))^{k_2}(f(3))^{k_3}\cdots}{k_1!\,k_2!\,k_3!\cdots}$$

where the summation is over all nonnegative $k_1, k_2, k_3, \cdots$ such that

$$(4.13)\quad n = k_1\cdot 1 + k_2\cdot 3 + k_3\cdot 5 + \cdots,\qquad k = k_1+k_2+k_3+\cdots\;.$$

Moreover, in view of the definition of $U(n,k)$, we have the following combinatorial interpretation of $W(n,k)$: $W(n,k)$ is the number of *weighted* number partitions (4.13): to each partition we assign the weight

$$\frac{n!}{(1!)^{k_1}(3!)^{k_2}(5!)^{k_3}\cdots}\;\frac{(f(1))^{k_1}(f(2))^{k_2}(f(3))^{k_3}\cdots}{k_1!\,k_2!\,k_3!}\;.$$

A different interpretation is suggested by (4.8).

5. We now return to Problem 1 as stated in the beginning of §2.

Let $T(n,k)$ denote the number of set partitions of Z_n into k blocks

$$B_1, B_2, \cdots B_k$$

of unequal length. Then it is evident that we have the generating function

(5.1)
$$\sum_{n=0}^{\infty} \sum_{k} T(n,k) \frac{x^n}{n!} z^k = \prod_{n=1}^{\infty} \left(1 + \frac{x^n z}{n!}\right) .$$

This is equivalent to

(5.2)
$$T(n,k) = \sum \frac{n!}{n_1! n_2! \cdots n_k!} ,$$

where the summation is over all $n_1, n_2, \cdots, n_k$ such that

(5.3)
$$n = n_1 + n_2 + \cdots + n_k, \qquad n_1 > n_2 > \cdots > n_k > 0.$$

In other words, $T(n,k)$ can be thought of as a weighted number partition: to each partition (5.3) we assign the weight

$$\frac{n!}{n_1! n_2! \cdots n_k!} ;$$

this weight is of course the number of admissible set partitions corresponding to the given number partition.

We can define a function that includes $T(n,k)$, $U(n,k)$, $V(n,k)$ as special cases. Let

(5.4)
$$\underline{r} = (r_1, r_2, r_3, \cdots)$$

be a sequence in which r_j is either a nonnegative integer or infinity. Let $S(n,k|\underline{r})$ denote the number of set partitions of Z_n into k blocks $B_1, B_2, \cdots, B_k$ with the requirement that, for each j, there are at most r_j blocks of length j. Thus, for example, we have

(5.5)
$$S(n,k|\underline{r}) = \begin{cases} S(n,k) & \underline{r} = (\infty, \infty, \infty, \cdots) \\ U(n,k) & \underline{r} = (\infty, 0, \infty, 0, \cdots) \\ V(n,k) & \underline{r} = (0, \infty, 0, \infty, \cdots) \\ T(n,k) & \underline{r} = (1, 1, 1, \cdots) \end{cases} .$$

For an arbitrary sequence (5.4) we have the generating function

(5.6)
$$\sum_{n=0}^{\infty} \sum_{k} S(n,k|\underline{r}) \frac{x}{n!} z^k = \prod_{j=1}^{\infty} \left\{ \sum_{k=0}^{r_j} \frac{1}{k!} \left(\frac{x^j z}{j!}\right)^k \right\} .$$

Clearly (5.6) reduces to a known result in each of the cases (5.5).

We shall now obtain some more explicit results for the enumerant $T(n,k)$. It is convenient to define

(5.7)
$$T_n(z) = \sum_{k} T(n,k) z^k$$

and

(5.8)
$$T_n = T_n(1) = \sum_{k} T(n,k) .$$

Then, by (5.1),

(5.9)
$$\sum_{n=0}^{\infty} T_n(z) \frac{x^n}{n!} = \prod_{n=1}^{\infty} \left(1 - \frac{x^n z}{n!}\right) .$$

Put

$$F = F(x,z) = \prod_{n=1}^{\infty} \left(1 + \frac{x^n z}{n!}\right) .$$

Then it is easily verified that

(5.10) $$\log F(x,z) = \sum_{n=1}^{\infty} F_n(z) \frac{x^n}{n!} ,$$

where

(5.11) $$F_n(z) = \sum_{rs=n} (-1)^{s-1} \frac{n!}{s(r!)^s} z^s .$$

Differentiating (5.10) with respect to x, we get

$$\frac{F_x(x,z)}{F(x,z)} = \sum_{n=0}^{\infty} F_{n+1}(z) \frac{x^n}{n!} .$$

This implies the recurrence

(5.12) $$T_{n+1}(z) = \sum_{r=0}^{n} \binom{n}{r} F_{r+1}(z) T_{n-r}(z) .$$

Differentiating (5.10) with respect to z, we get

$$\frac{F_z(x,z)}{F(x,z)} = \sum_{n=1}^{\infty} F'_n(z) \frac{x^n}{n!}$$

and therefore

(5.13) $$T'_n(z) = \sum_{r=1}^{n} \binom{n}{r} F'_n(z) T_{n-r}(z).$$

Written at length, (5.13) becomes

(5.14) $$\sum_k kT(n,k)z^k = \sum_{r=1}^{n} \binom{n}{r} T(n-r,j) \sum_{st=r} (-1)^{s-1} \frac{r!}{(t!)^s} z^s .$$

This gives

(5.15) $$kT(n,k) = \sum_{\substack{0<st\leqslant n \\ s\leqslant t}} (-1)^{s-1} \binom{n}{st} \frac{(st)!}{(t!)^s} T(n-st, k-s) .$$

It is obvious that

(5.16) $$T(n,1) = 1 \qquad (n \geqslant 1).$$

Using (5.14) we get

(5.17) $$T(n,2) = \tfrac{1}{2}(2^n - 2) - \tfrac{1}{2} \binom{n}{n/2} = S(n,2) - \tfrac{1}{2} \binom{n}{n/2} .$$

If we put

(5.18) $$G_k(x) = \sum_n T(n,k) \frac{x^n}{n!}$$

and

(5.19) $$H_j(x) = \sum_{t=1}^{\infty} \frac{x^{jt}}{(t!)^j} ,$$

then by (5.14)

$$kG_k(x) = \sum_{s=1}^{\infty} (-1)^{s-1} H_s(x) G_{k-s}(x). \tag{5.20}$$

Thus for example

$G_1(x) = H_1(x) = e^x - 1, \qquad 2!G_2(x) = H_1^2(x) - H_2(x), \qquad 3!G_3(x) = H_1^3(x) - 3H_1(x)H_2(x) + 2H_3(x)$

and so on.

If we take $z = 1$ in (5.12) we get the recurrence

$$T_{n+1} = \sum_{r=0}^{\infty} \binom{n}{r} F_{r+1}(1) T_{n-r} . \tag{5.21}$$

Unfortunately the numbers

$$F_n(1) = \sum_{rs=n} (-1)^{s-1} \frac{n!}{s(r!)^s}$$

are not simple. We note that

$$\sum_{n=1}^{\infty} F_n(1) \frac{x^n}{n!} = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s} H_s(x). \tag{5.22}$$

Analogous to (5.2) we may define

$$T_1(n,k) = \sum \frac{n!}{n_1 n_2 \cdots n_k} , \tag{5.23}$$

where again the summation is over all $n_1, n_2, \cdots, n_k$ such that

$$n = n_1 + n_2 + \cdots + n_k , \qquad n_1 > n_2 > \cdots > n_k > 0.$$

Then $T_1(n,k)$ denotes the number of permutations of Z_n with k cycles of unequal length. From (5.23) we obtain the generating function

$$\sum_{n=0}^{\infty} \sum_{k} T_1(n,k) \frac{x^n}{n!} z^k = \prod_{n=1}^{\infty} \left(1 + \frac{x^n z}{n} \right) . \tag{5.24}$$

As above we define

$$T_{1,n}(z) = \sum_k T_1(n,k) z^k, \qquad T_{1,n} = T_{1,n}(1) = \sum_k T_1(n,k).$$

We can obtain recurrences for $T_1(n,k)$ and $T_{1,n}$ similar to those for $T(n,k)$ and T_n. In particular we have

$$T_{1,n+1} = \sum_{r=0}^{n} \binom{n}{r} F_{1,r+1}(1) T_{1,n-r} , \tag{5.25}$$

where

$$F_{1,n}(1) = \sum_{rs=n} (-1)^{s-1} \frac{n!}{sr^s} .$$

We remark that $T_{1,n}$ is the total number of permutations of Z_n with cycles of unequal length. Note that

$$\sum_{n=1}^{\infty} T_{1,n} \frac{x^n}{n!} = \prod_{n=1}^{\infty} \left(1 + \frac{x^n}{n} \right). \tag{5.26}$$

Finally, as in (5.4), let

$$\underset{\sim}{r} = (r_1, r_2, r_3, \cdots) \tag{5.27}$$

be a sequence in which each r_j is either a nonnegative integer or infinity. Let $S_1(n,k|\underline{r})$ denote the number of permutations π in Z_n with the requirement that, for each i, the number of cycles of length i in π is at most r_i. Then

$$S_1(n,k|\underline{r}) = \begin{cases} S_1(n,k) & \underline{r} = (\infty, \infty, \infty, \cdots) \\ U_1(n,k) & \underline{r} = (\infty, 0, \infty, 0, \cdots) \\ V_1(n,k) & \underline{r} = (0, \infty, 0, \infty, \cdots) \\ T_1(n,k) & \underline{r} = (1, 1, 1, \cdots). \end{cases}$$

For an arbitrary sequence (5.27) we have the generating function

$$\sum_{n=0}^{\infty} \sum_k S_1(n,k|\underline{r}) \frac{x^n}{n!} z^k = \prod_{j=1}^{\infty} \left\{ \sum_{k=0}^{r_j} \frac{1}{k!} \left(\frac{x^j z}{j}\right)^k \right\}. \tag{5.28}$$

The following question is of some interest. For what sequences (5.27) will the orthogonality relations

$$\sum_{j=k}^{n} (-1)^{n-j} S_1(n,j|\underline{r}) S(j,k|\underline{r}) \tag{5.29}$$

$$= \sum_{j=k}^{n} (-1)^{j-k} S(n,j|\underline{r}) S_1(j,k|\underline{r}) = \delta_{n,k}$$

be satisfied?

Alternatively we may ask for what pairs of sequences $\underline{r}$, $\underline{s}$ will the orthogonality relations

$$\sum_{j=k}^{n} (-1)^{n-j} S_1(n,j|\underline{r}) S(j,k|\underline{s}) = \sum_{j=k}^{n} (-1)^{j-k} S(n,j|\underline{s}) S_1(j,k|\underline{r}) = \delta_{n,k} \tag{5.30}$$

be satisfied?

REFERENCES

1. L. Carlitz, "Single Variable Bell Polynomials," *Collectanea Mathematica,* Vol. 14 (1962), pp. 13–25.
2. L. Carlitz and John Riordan, "The Divided Central Differences of Zero," *Canadian Journal of Mathematics,* Vol. 15 (1963), pp. 94–100.
3. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* Clarendon Press, Oxford, 1938.
4. John Riordan, *An Introduction to Combinatorial Analysis,* Wiley, New York, 1958.
5. Gian-Carlo Rota, "The Number of Partitions of a Set," *American Math. Monthly,* Vol. 71 (1964), pp. 498–504.

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