SOME SUMS OF MULTINOMIAL COEFFICIENTS

L. CARLITZ *
Duke University, Durham, North Carolina 27706

1. Recent interest in some lacunary sums of binomial coefficients (see for example [2], [3]) suggests that it may be of interest to consider some simple sums of multinomial coefficients.

Put

$$(i,j,k) = \frac{(i+j+k)!}{i!\,j!\,k!}\ ,$$

so that

$$(x+y+z)^n = \sum_{i+j+k=n} (i,j,k)x^iy^jz^k\ . \tag{1.1}$$

Let $\epsilon_1,\ \epsilon_2,\ \epsilon_3\ = \pm 1$ and define

$$S_{000} = S_{000}(n) = \sum_{i,j,k \text{ even}} (i,j,k)\,,$$

$$S_{100} = S_{100}(n) = \sum_{\substack{i \text{ odd}\\ j,k \text{ even}}} (i,j,k), \text{ etc.,}$$

where in each case the summation is over all non-negative $i,\ j,\ k$ such that $i+j+k=n$. Since

$$S_{100} = S_{010} = S_{001},\ S_{011} = S_{101} = S_{110}\ ,$$

it is evident from (1.1) that

$$S_{000} + S_{100}(\epsilon_1+\epsilon_2+\epsilon_3) + S_{011}(\epsilon_2\epsilon_3+\epsilon_3\epsilon_1+\epsilon_1\epsilon_2) + S_{111}\epsilon_1\epsilon_2\epsilon^3 = (\epsilon_1+\epsilon_2+\epsilon_3)^n\,. \tag{1.2}$$

Specializing the ϵ_j we get

$$(1,1,1): S_{000} + 3S_{100} + 3S_{110} + S_{111} = 3^n$$

$$(-1,1,1) = S_{000} + S_{100} - S_{110} - S_{111} = 1$$

$$(1,-1,-1): S_{000} - S_{100} - S_{110} + S_{111} = (-1)^n$$

$$(-1,-1,-1): S_{000} - 3S_{100} + 3S_{110} - S_{111} = (-3)^3\,.$$

Solving for the S_{ijk}, we get

$$\begin{cases} 8S_{000} = 3^n + 3 + 3(-1)^n + (-3)^n \\ 8S_{100} = 3^n + 1 - (-1)^n - (-3)^n \\ 8S_{110} = 3^n - 1 - (-1)^n + (-3)^n \\ 8S_{111} = 3^n - 3 + 3(-1)^n - (-3)^n\ . \end{cases} \tag{1.3}$$

Tabulating even and odd values of n separately, (1.3) reduces to

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$$(1.4)\qquad \begin{cases} S_{000}(2n) = \tfrac14(3^{2n}+3)\\ S_{100}(2n) = 0\\ S_{110}(2n) = \tfrac14(3^{2n}-1)\\ S_{111}(2n) = 0\end{cases}$$

$$(1.5)\qquad \begin{cases} S_{000}(2n+1) = 0\\ S_{100}(2n+1) = \tfrac14(3^{2n+1}+1)\\ S_{110}(2n+1) = 0\\ S_{111}(2n+1) = \tfrac14(3^{2n+1}-3).\end{cases}$$

It follows from (1.4) and (1.5) that

$$(1.6)\qquad S_{000}(2n) = S_{110}(2n)+1, \qquad S_{100}(2n+1) = S_{111}(2n+1)+1.$$

We also have the generating functions

$$(1.7)\qquad \begin{cases} \sum_{n=0}^{\infty} S_{000}(n)x^n = \tfrac14\left(\frac{1}{1-9x^2}+\frac{3}{1-x^2}\right) = \frac{1-7x^2}{(1-x^2)(1-9x^2)}\\ \sum_{n=0}^{\infty} S_{100}(n)x^n = \tfrac14\left(\frac{3x}{1-9x^2}+\frac{x}{1-x^2}\right) = \frac{1-3x^2}{(1-x^2)(1-9x^2)}\\ \sum_{n=0}^{\infty} S_{110}(n)x^n = \tfrac14\left(\frac{1}{1-9x^2}-\frac{1}{1-x^2}\right) = \frac{2x^2}{(1-x^2)(1-9x^2)}\\ \sum_{n=0}^{\infty} S_{111}(n)x^n = \tfrac14\left(\frac{3x}{1-9x^2}-\frac{3x}{1-x^2}\right) = \frac{6x^3}{(1-x^2)(1-9x^2)}\end{cases}.$$

2. Let $m > 1$ and define

$$(2.1)\qquad S_{ijk} = S_{ijk}^{(m)}(n) = \sum_{r+s+t=n} (r,s,t),$$

where the summation is restricted to non-negative r, s, t such that

$$r \equiv i, \quad s \equiv j, \quad t \equiv k \pmod{m}.$$

We may also assume that

$$0 \leqslant i < m, \quad 0 \leqslant j < m, \quad 0 \leqslant k < m.$$

Clearly S_{ijk} is symmetric in the three indices i, j, k. Also it is evident from the definition that

$$(2.2)\qquad S_{ijk}^{(m)}(n) = 0 \quad (n \not\equiv i+j+k \pmod{m}).$$

Hence in what follows it will suffice to assume that $n \equiv i+j+k \pmod{m}$.

Let ζ denote a fixed primitive m^{th} root of 1. Then it is clear from (1.1) and (2.1), that for arbitrary integers, $a, b, c,$

$$(2.3)\qquad (\zeta^a+\zeta^b+\zeta^c) = \sum_{r+s+t=n} \zeta^{ra+sb+tc}(r,s,t) = \sum_{i,j,k=0}^{m-1} \zeta^{ia+jb+kc} S_{ijk}^{(m)}(n).$$

Since

$$\sum_{a=0}^{m-1} \zeta^{ra} = \begin{cases} m & (r=0) \\ 0 & (0<r<m), \end{cases}$$

it follows from (2.3) that

$$m^3 S_{ijk}^{(m)}(n) = \sum_{a,b,c=0}^{m-1} (\zeta^a+\zeta^b+\zeta^c)^n \zeta^{-ai-bj-ck}. \tag{2.4}$$

While this theoretically evaluates $S_{ijk}^{(m)}(n)$, it is not really satisfactory. For $m = 3$ more explicit results are obtainable without a great deal of computation.

By (2.4) we have

$$27S_{000}^{(3)}(n) = \sum_{a,b,c=0}^{2} (\omega^a+\omega^b+\omega^c)^3,$$

where $\omega^2+\omega+1 = 0$. This reduces to

$$27S_{000}^{(3)}(n) = 3^n + 3(\omega+2)^n + 3(\omega^2+2)^n + 3(2\omega+1)^n + 3(2\omega^2+1)^n$$
$$+ 3(2\omega^2+\omega)^n + 3(2\omega+\omega^2)^n + 6(\omega^2+\omega+1)^n + (3\omega)^n + (3\omega^2)^n .$$

By (2.2),

$$S_{000}^{(3)}(n) = 0 \quad (n \not\equiv 0 \pmod 3).$$

For n a multiple of 3 we get, replacing n by $3n$,

$$27S_{000}^{(3)}(3n) = 3^{3n+1} + 9(2\omega+1)^{3n} + 9(2\omega^2+1)^{3n} \quad (n>0).$$

This reduces to

$$\begin{cases} S_{000}^{(3)}(6n) = 3^{6n-2} + 2(-1)^n 3^{3n-1} & (n>0) \\ S_{000}^{(3)}(6n+3) = 3^{6n+1} & (n \geq 0). \end{cases} \tag{2.5}$$

Check.

$$S_{000}^{(3)}(6) = \frac{3\cdot 6!}{6!\,0!\,0!} + \frac{3\cdot 6!}{3!\,3!\,0!} = 3 + 3\cdot 60 = 63 = 3^4 - 2\cdot 3^2, \quad S_{000}^{(3)}(3) = \frac{3\cdot 3!}{3!\,0!\,0!} = 3,$$

$$S_{000}^{(3)}(9) = \frac{9!}{3!\,3!\,3!} + \frac{6\cdot 9!}{6!\,3!\,0!} + \frac{3\cdot 9!}{9!\,0!\,0!} = 5\cdot 6\cdot 7\cdot 8 + 7\cdot 8\cdot 9 + 3 = 3\cdot 729 = 3^7 .$$

Similarly we have

$$27S_{111}^{(3)}(n) = \sum_{a,b,c=0}^{2} (\omega^a+\omega^b+\omega^c)^n \omega^{-a-b-c} = 3^n + 3(\omega+2)^n\omega^{-1} + 3(\omega^2+2)^n\omega^{-2}$$
$$+ 3(2\omega+1)^n\omega^{-2} + 3(2\omega^2+1)^n\omega^{-1} + 3(2\omega^2+\omega)^n\omega^{-2} + 3(2\omega+\omega^2)\omega^{-1}$$
$$+ 6(\omega^2+\omega+1)^n + (3\omega)^n + (3\omega^2)^n .$$

As in the previous case,

$$S_{111}^{(3)}(n) = 0 \quad (n \not\equiv 0 \pmod 3),$$

while

$$27S_{111}^{(3)}(3n) = 3\cdot 3^{3n} + 3(2\omega^2+1)^{3n}\omega^{-1} + 3(2\omega+1)^{3n}\omega^{-2} + 3(2\omega+1)^{3n}\omega^{-2} + 3(2\omega^3+1)^{3n}\omega^{-1}$$
$$+ 3(2\omega+1)^{3n}\omega^{-2} + 3(2\omega^2+1)^{3n}\omega^{-1} + 6(\omega^2+\omega+1)^{3n}$$
$$= 3^{3n+1} + 9(\sqrt{-3})^{3n}\omega^{-2} + 9(-\sqrt{-3})^{3n}\omega^{-1} + 6(\omega^2+\omega+1)^{3n}.$$

It follows that

$$\begin{cases} S_{111}^{(3)}(6n) = 3^{6n-2} - (-1)^n 3^{3n-1} & (n>0) \\ S_{111}^{(3)}(6n+3) = 3^{6n+1} + (-1)^n 3^{3n+1} & (n \geq 0). \end{cases} \tag{2.6}$$

Check.

$$S^{(3)}_{111}(6) = \frac{3\cdot 6!}{4!\,1!\,1!} = 3\cdot 6\cdot 5 = 90 = 3^4 + 3^2, \qquad S^{(3)}_{111}(3) = \frac{3!}{1!\,1!\,1!} = 6 = 3+3,$$

$$S^{(3)}_{111}(9) = \frac{3\cdot 9!}{7!\,1!\,1!} + \frac{3\cdot 9!}{4!\,4!\,1!} = 3\cdot 8\cdot 9 + 3\cdot 9\cdot 70 = 3^4\cdot 26 = 3^7 - 3^4.$$

We find also that

$$(2.7) \qquad \begin{cases} S^{(3)}_{222}(6n) = 3^{6n-2} - (-1)^n 3^{3n-1} & (n > 0) \\ S^{(3)}_{222}(6n+3) = 3^{6n+1} - (-1)^n 3^{3n+1} & (n \geq 0). \end{cases}$$

Check.

$$S^{(3)}_{222}(6) = \frac{6!}{2!\,2!\,2!} = 90 = 3^4 + 3^2, \qquad S^{(3)}_{222}(9) = \frac{3\cdot 9!}{5!\,2!\,2!} = 3^4\cdot 28 = 3^7 + 3^4.$$

Note that it follows from (2.6) and (2.7) that

$$(2.8) \qquad S^{(3)}_{111}(6n) = S^{(3)}_{222}(6n)$$

and from (2.5), (2.6), (2.7),

$$(2.9) \qquad S^{(3)}_{111}(6n+3) + S^{(3)}_{222}(6n+3) = 2S^{(3)}_{000}(6n+3).$$

3. Since

$$(r, s, t) = (r-1, s, t) + (r, s-1, t) + (r, s, t-1),$$

it follows from (2.1) that

$$(3.1) \qquad S^{(m)}_{i,j,k}(n) = S^{(m)}_{i-1,j,k}(n-1) + S^{(m)}_{i,j-1,k}(n-1) + S^{(m)}_{i,j,k-1}(n-1),$$

where

$$S^{(m)}_{i,j,k}(n) = S^{(m)}_{i',j',k'}(n)$$

when

$$i \equiv i', \quad j \equiv j', \quad k \equiv k' \pmod m.$$

In particular

$$(3.2) \qquad S^{(m)}_{i,i,i}(n) = 3S^{(m)}_{i,i,i-1}(n-1).$$

For example, when $m = 2$, we have

$$\begin{cases} S^{(2)}_{000}(2n) = 3S^{(2)}_{100}(2n-1) \\ S^{(2)}_{111}(2n+1) = 3S^{(2)}_{110}(2n) \\ S^{(2)}_{100}(2n+1) = S^{(2)}_{000}(2n) + 2S^{(2)}_{110}(2n) \\ S^{(2)}_{110}(2n) = S^{(2)}_{111}(2n-1) + 2S^{(2)}_{100}(2n-1) \end{cases}$$

in agreement with previous results.

The case $m = 3$ is more interesting as well as more involved. We have, to begin with,

$$\begin{cases} S^{(3)}_{000}(3n) = 3S^{(3)}_{200}(3n-1) \\ S^{(3)}_{111}(3n) = 3S^{(3)}_{110}(3n-1) \\ S^{(3)}_{222}(3n) = 3S^{(3)}_{221}(3n-1). \end{cases}$$

It therefore follows from (2.5), (2.6), and (2.7) that

$$(3.3) \qquad \begin{cases} S^{(3)}_{200}(6n+5) = 3^{6n+3} - 2(-1)^n 3^{3n+1} \\ S^{(3)}_{200}(6n+2) = 3^{6n} \end{cases} \qquad (n \geq 0)$$

(3.4)
$$\begin{cases} S^{(3)}_{110}(6n+5) = 3^{6n+3} + (-1)^n 3^{3n+1} \\ S^{(3)}_{110}(6n+2) = 3^{6n} + (-1)^n 3^{3n} \end{cases} \qquad (n \geq 0)$$

(3.5)
$$\begin{cases} S^{(3)}_{221}(6n+5) = 3^{6n+3} + (-1)^n 3^{3n+1} \\ S^{(3)}_{221}(6n+2) = 3^{6n} - (-1)^n 3^{3n} \end{cases} \qquad (n \geq 0).$$

Check.

$$S^{(3)}_{200}(5) = \frac{5!}{5!\,0!\,0!} + \frac{2\cdot 5!}{2!\,3!\,0!} = 1 + 20 = 21 = 3^3 - 2\cdot 3,$$

$$S^{(3)}_{200}(8) = \frac{8!}{8!\,0!\,0!} + \frac{2\cdot 8!}{5!\,3!\,0!} + \frac{2\cdot 8!}{2!\,6!\,0!} + \frac{8!}{2!\,3!\,3!}$$
$$= 1 + 112 + 56 + 560 = 729 = 3^6;$$

$$S^{(3)}_{110}(5) = \frac{5!}{1!\,1!\,3!} + \frac{2\cdot 5!}{4!\,1!\,0!} = 20 + 10 = 30 = 3^3 + 3,$$

$$S^{(3)}_{110}(8) = \frac{2\cdot 8!}{7!\,1!\,0!} + \frac{8!}{4!\,4!\,0!} + \frac{2\cdot 8!}{1!\,4!\,3!} + \frac{8!}{1!\,1!\,6!} = 16 + 70 + 560 + 56 = 702 = 3^6 - 3^3;$$

$$S^{(3)}_{221}(5) = \frac{5!}{2!\,2!\,1!} = 30 = 3^3 + 3,$$

$$S^{(3)}_{221}(8) = \frac{2\cdot 8!}{5!\,2!\,1!} + \frac{8!}{2!\,2!\,4!} = 27\cdot 28 = 3^6 + 3^3.$$

In the next place, it follows from

$$S^{(3)}_{210}(3n) = S^{(3)}_{110}(3n-1) + S^{(3)}_{200}(3n-1) + S^{(3)}_{221}(3n-1)$$

that

$$S^{(3)}_{210}(6n) = S^{(3)}_{110}(6n-1) + S^{(3)}_{200}(6n-1) + S^{(3)}_{221}(6n-1)$$
$$= (3^{6n-3} - (-1)^n 3^{3n-2}) + (3^{6n-3} + 2(-1)^n 3^{3n-2}) + (3^{6n-3} - (-1)^n 3^{3n-2}) = 3^{6n-2} \quad (n > 0).$$

$$S^{(3)}_{210}(6n+3) = S^{(3)}_{110}(6n+2) + S^{(3)}_{200}(6n+2) + S^{(3)}_{221}(6n+2)$$
$$= (3^{6n} + (-1)^n 3^{3n}) + 3^{6n} + (3^{6n} - (-1)^n 3^{3n}) = 3^{6n+1} \qquad (n \geq 0)$$

that is,

(3.6)
$$\begin{cases} S^{(3)}_{210}(6n) = 3^{6n-2} & (n > 0) \\ S^{(3)}_{210}(6n+3) = 3^{6n+1} & (n \geq 0). \end{cases}$$

Check.

$$S^{(3)}_{210}(6) = \frac{6!}{2!\,1!\,3!} + \frac{6!}{5!\,1!\,0!} + \frac{6!}{2!\,4!\,0!} = 60 + 6 + 15 = 3^4,$$

$$S^{(3)}_{210}(3) = \frac{3!}{2!\,1!\,0!} = 3,$$

$$S^{(3)}_{210}(9) = \frac{9!}{2!\,1!\,6!} + \frac{9!}{5!\,1!\,3!} + \frac{9!}{2!\,4!\,3!} + \frac{9!}{8!\,1!\,0!} + \frac{9!}{5!\,4!\,0!} + \frac{9!}{2!\,7!\,0!}$$
$$= 9\cdot 4\cdot 7 + 9\cdot 8\cdot 7 + 9\cdot 7\cdot 20 + 9 + 9\cdot 7\cdot 2 + 9\cdot 4 = 9\cdot 243 = 3^7.$$

Next it follows in like manner from

$$\begin{cases} S^{(3)}_{211}(3n+1) = S^{(3)}_{111}(3n) + 2S^{(3)}_{210}(3n) \\ S^{(3)}_{220}(3n+1) = S^{(3)}_{222}(3n) + 2S^{(3)}_{210}(3n) \\ S^{(3)}_{100}(3n+1) = S^{(3)}_{000}(3n) + 2S^{(3)}_{210}(3n) \end{cases}$$

that

(3.7) $$\begin{cases} S^{(3)}_{211}(6n+1) = 3^{6n-1} - (-1)^n 3^{3n-1} \\ S^{(3)}_{211}(6n+4) = 3^{6n+2} + (-1)^n 3^{n+1} \end{cases} \quad (n \geq 0)$$

(3.8) $$\begin{cases} S^{(3)}_{220}(6n+1) = 3^{6n-1} - (-1)^n 3^{3n-1} \\ S^{(3)}_{220}(6n+4) = 3^{6n+2} - (-1)^n 3^{3n+1} \end{cases} \quad (n \geq 0)$$

(3.9) $$\begin{cases} S^{(3)}_{100}(6n+1) = 3^{6n-1} + 2(-1)^n 3^{3n-1} \\ S^{(3)}_{100}(6n+4) = 3^{6n+2} \end{cases} \quad (n \geq 0).$$

Check.

$$S^{(3)}_{211}(7) = \frac{7!}{5!\,1!\,1!} + \frac{2!\,7!}{2!\,4!\,1!} = 42 + 210 = 252 = 3^5 + 3^2, \quad S^3_{211}(4) = \frac{4!}{2!\,1!\,1!} = 12 = 3^2 + 3,$$

$$S^{(3)}_{211}(10) = \frac{10!}{8!\,1!\,1!} + \frac{2\cdot 10!}{5!\,4!\,1!} + \frac{10!}{2!\,4!\,4!} + \frac{2\cdot 10!}{2!\,7!\,1!}$$
$$= 9\cdot 10 + 9\cdot 280 + 9\cdot 350 + 9\cdot 80 = 3^4\cdot 80 = 3^8 - 3^4;$$

$$S^{(3)}_{220}(7) = \frac{7!}{2!\,2!\,3!} + \frac{2\cdot 7!}{5!\,2!\,0!} = 3^2\cdot 28 = 3^5 + 3^2,$$

$$S^{(3)}_{220}(4) = \frac{4!}{2!\,2!\,0!} = 6 = 3^2 - 3,$$

$$S^{(3)}_{220}(10) = \frac{10!}{2!\,2!\,6!} + \frac{2\cdot 10!}{5!\,2!\,3!} + \frac{10!}{5!\,5!\,0!} + \frac{2\cdot 10!}{8!\,2!\,0!} = 9^2\cdot 82 = 3^8 + 3^4.$$

$$S^{(3)}_{100}(7) = \frac{7!}{7!\,0!\,0!} + \frac{2\cdot 7!}{4!\,3!\,0!} + \frac{7!}{1!\,3!\,3!} + \frac{2\cdot 7!}{1!\,6!\,0!} = 1 + 70 + 140 + 14 = 3^2\cdot 25 = 3^5 - 2\cdot 3^2,$$

$$S^{(3)}_{100}(4) = \frac{4!}{4!\,0!\,0!} + \frac{2\cdot 4!}{1!\,3!\,0!} = 1 + 8 = 3^2,$$

$$S^{(3)}_{100}(10) = \frac{10!}{10!\,0!\,0!} + \frac{2\cdot 10!}{7!\,3!\,0!} + \frac{10!}{4!\,3!\,3!} + \frac{2\cdot 10!}{4!\,6!\,0!} + \frac{2\cdot 10!}{1!\,9!\,0!} + \frac{2\cdot 10!}{1!\,6!\,3!}$$
$$= 1 + 240 + 4200 + 420 + 20 + 1680 = 3^8.$$

This completes the evaluation of the ten functions $S^{(3)}_{ijk}(n)$.

4. The five functions $S^{(2)}_{ijk\ell}(n)$ can be evaluated without much computation. To begin with, we have

$$2^4 S^{(2)}_{0000}(2n) = (1+1+1+1)^{2n} + 4(1+1+1-1)^{2n} + 6(1+1-1-1)^{2n}$$
$$+ 4(1-1-1-1)^{2n} + (-1-1-1-1)^{2n},$$

which reduces to

(4.1) $$S^{(2)}_{000}(2n) = 2^{4n-3} + 2^{2n-1} \quad (n > 0).$$

Since

$$S^{(2)}_{0000}(2n) = 4S^{(2)}_{1000}(2n-1),$$

we get

(4.2) $$S^{(2)}_{1000}(2n+1) = 2^{4n-1} + 2^{2n-1} \quad (n \geq 0).$$

Next, since

$$S^{(2)}_{1000}(2n+1) = S^{(2)}_{0000}(2n) + 3S^{(2)}_{1100}(2n),$$

it follows that

(4.3) $$S^{(2)}_{1100}(2n) = 2^{4n-3} \quad (n > 0).$$

Similarly, from

$$S^{(2)}_{1100}(2n) = 2S^{(2)}_{1000}(2n-1) + 2S^{(2)}_{1110}(2n-1)$$

we get

(4.4) $$S^{(2)}_{1110}(2n+1) = 2^{4n-1} - 2^{2n-1} \quad (n \geq 0).$$

Finally, it follows from

$$S^{(2)}_{1110}(2n+1) = S^{(2)}_{1111}(2n) + 3S^{(2)}_{1100}(2n)$$

that

(4.5) $$S^{(2)}_{1111}(2n) = 2^{4n-3} - 2^{2n-1} \qquad (n \geq 1).$$

For example

$$S^{(2)}_{1111}(6) = \frac{4\cdot 6!}{3!\,1!\,1!\,1!} = 480 = 2^9 - 2^5.$$

Note that it follows from the above results that

(4.6) $$S^{(2)}_{0000}(2n) + S^{(2)}_{1111}(2n) = 2S^{(2)}_{1100}(2n)$$

and

(4.7) $$S^{(2)}_{1000}(2n+1) + S^{(2)}_{1110}(2n+1) = 8S^{(2)}_{1100}(2n).$$

5. The results of §4 suggest that it would be of interest to evaluate

(5.1) $$f_{j,k}(n) = S^{(2)}_{\underbrace{1\cdots1}_{j}\underbrace{0\cdots0}_{k}}(n),$$

where j, k are arbitrary non-negative integers and the right-hand side of (5.1) has the obvious meaning. Clearly

(5.2) $$f_{j,k}(n) = 0 \qquad (n \not\equiv j \pmod 2).$$

To begin with, we have

$$\begin{aligned}2^k f_{0,k}(n) &= (1+1+\cdots+1)^n + \binom{k}{1}(1+\cdots+1-1)^n \\ &\quad + \binom{k}{2}(1+\cdots+1-1-1)^n + \cdots + \binom{k}{k}(-1-1-\cdots-1)^n \\ &= k^n + \binom{k}{1}(k-2)^n + \binom{k}{2}(k-4)^n + \cdots + \binom{k}{k}(-k)^n.\end{aligned}$$

Thus

(5.3) $$f_{0,k}(n) = 2^{-k}\sum_{j=0}^{k}\binom{k}{j}(k-2j)^n.$$

Since

(5.4) $$f_{0,k}(n) = kf_{1,k-1}(n-1),$$

it follows at once that $S_{1,k-1}$ can be evaluated explicitly by means of (5.3). Next, since

$$f_{1,k-1}(n-1) = f_{0,k}(n-2) + (k-1)f_{2,k-2}(n-2),$$

we get

(5.5) $$k(k-1)f_{2,k-2}(n-2) = f_{0,k}(n) = kf_{0,k}(n-2).$$

Similarly it follows from

$$f_{2,k-2}(n-2) = 2f_{1,k-1}(n-3) + (k-2)f_{3,k-3}(n-3)$$

that

(5.6) $$k(k-1)(k-2)f_{3,k-3}(n-3) = f_{0,k}(n) - (3k-2)f_{0,k}(n-2).$$

We also find that

(5.7) $$k(k-1)(k-2)(k-3)f_{4,k-4}(n-4) = f_{0,k}(n) - 2(3k-4)f_{0,k}(n-2) + 3k(k-2)f_{0,k}(n-4),$$

(5.8) $$\begin{aligned}k(k-1)(k-2)(k-4)f_{5,k-5}(n-5) &= f_{0,k}(n) - 2(5k-10)f_{0,k}(n-2) \\ &\quad + (15k^2 - 50k + 24)f_{0,k}(n-4).\end{aligned}$$

These results suggest the following general formula:

(5.9) $$\frac{k!}{(k-j)!}f_{j,k-j}(n-j) = \sum_{2s\leq j}(-1)^s P_{j,s}(k)f_{0,k}(n-2s) \qquad (0 \leq j \leq k),$$

where $P_{j,s}(k)$ denotes a polynomial in k of degree s. Since

$$f_{j,k-j}(n-j) = jf_{j-1,k-j+1}(n-j-1) + (k-j)f_{j+1,k-j-1}(n-j-1),$$

it follows that

$$\frac{(k-j)!}{k!}\sum_{2s\leqslant j}(-1)^s P_{j,s}(k)f_{0,k}(n-2s) = j\,\frac{(k-j+1)!}{k!}\sum_{2s<j}(-1)^s P_{j-1,s}(k)f_{0,k}(n-2s-2)$$

$$+ (k-j)\,\frac{(k-j-1)!}{k!}\sum_{2s\leqslant j+1}(-1)^s P_{j+1,s}(k)f_{0,k}(n-2s).$$

Hence we take

(5.10) $$P_{j+1,s}(k) = P_{j,s}(k) + j(k-j+1)P_{j-1,s-1}(k).$$

$P_{j,s}(k)$				
$j \backslash s$	0	1	2	3
0	1			
1	1			
2	1	k		
3	1	$3k-2$		
4	1	$6k-8$	$3k(k-2)$	
5	1	$10k-20$	$15k^2-50k+24$	
6	1	$15k-40$	$45k^2-210k+184$	$15k(k-2)(k-4)$
7	1	$21k-70$	$105k^2-630k+784$	$105k^3-840k^2+1764k-720$

It is evident that

(5.11) $$P_{j,0}(k) = 1 \qquad (j \geqslant 0).$$

Also it follows easily from (5.10) that

(5.12) $$P_{2j,j}(k) = 1\cdot 3\cdot 5\cdots(2j-1)k(k-2)(k-4)\cdots(k-2j+2) \qquad (j \geqslant 0).$$

Since

$$P_{j+1,1}(k) = P_{j,1}(k) + j(k-j+1) \qquad (j \geqslant 1),$$

we get

$$P_{j+1,1}(k) = \sum_{t=1}^{j} t(k-t+1).$$

This gives

(5.13) $$P_{j,1}(k) = \frac{1}{2}j(j-1)k - \frac{1}{3}j(j-1)(j-2) \qquad (j \geqslant 0).$$

Similarly, since

$$P_{j+1,2}(k) = P_{j,2}(k) + j(k-j+1)P_{j-1,1}(k) = P_{j,2}(k) + \tfrac{1}{2}j(j-1)(j-2)k^2 - \left\{\tfrac{5}{6}j(j-1)(j-2)(j-3) + j(j-1)(j-2)\right\}k + \tfrac{1}{3}j(j-1)(j-2)(j-3)(j-4) + j(j-1)(j-2)(j-3),$$

we find that

$$(5.14)\qquad P_{j,2}(k) = 3\binom{j}{4}k^2 - \left[20\binom{j}{5}+6\binom{j}{4}\right]k + \left[40\binom{j}{6}+24\binom{j}{5}\right]$$
$$= \frac{1}{8}j(j-1)(j-2)(j-3)k^2 - \frac{1}{12}j(j-1)(j-2)(j-3)(2j-5)k^2$$
$$+\frac{1}{90}j(j-1)(j-2)(j-3)(j-4)(5j-7).$$

For example

$$P_{6,2}(k) = 3\cdot 15k^2 - (20\cdot 6+6\cdot 15)k + (40+24\cdot 6) = 45k^2 - 210k + 184.$$

We also find that

$$(5.15)\qquad P_{j,3}(k) = 15\binom{j}{6}k^3 - \left[210\binom{j}{7}+90\binom{j}{6}\right]k^2$$
$$+\left[1120\binom{j}{8}+924\binom{j}{7}+120\binom{j}{6}\right]k - \left[2240\binom{j}{9}+2688\binom{j}{8}+720\binom{j}{7}\right]$$

For example

$$P_{7,3}(k) = 15\cdot 7k^3 - (210+90\cdot 7)k^2 + (924+120\cdot 7) - 720 = 105k^3 - 840k^2 + 1764k - 720.$$

We have noted above that $P_{j,s}(k)$ is a polynomial in k of degree s. In addition we can assert that $P_{j,s}(k)$ i polynomial in j of degree $3s$. More precisely, if we put

$$(5.16)\qquad P_{j,s}(k) = \sum_{t=0}^{s}(-1)^s c_{s,t}(j)k^{s-t},$$

then $c_{s,t}(j)$ is a polynomial in j of degree $2s+t$. If we substitute from (4.7) in (4.1) we get

$$\sum_{t=0}^{s}(-1)^t[c_{s,t}(j+1)-c_{s,t}(j)]k^{s-t} = j(k-j+1)\sum_{t=0}^{s-1}(-1)^t c_{s-1,t}(j-1)k^{s-t-1}.$$

This gives

$$(5.17)\qquad c_{s,t}(j+1)-c_{s,t}(j) = jc_{s-1,t}(j-1)+j(j-1)c_{s-1,t-1}(j-1).$$

The table of values of $P_{j,s}(k)$ suggests that

$$(5.18)\qquad \begin{cases} \displaystyle\sum_{s=0}^{j}(-1)^{j-s}P_{2j,s}(k) = 1\cdot 3\cdot 5\cdots(2j-1)(k-1)(k-3)\cdots(k-2j+1) \\ \displaystyle\sum_{s=0}^{j}(-1)^{j-s}P_{2j+1,s}(k) = 1\cdot 3\cdot 5\cdots(2j-1)(2j+1)\cdot(k-1)(k-3)\cdots(k-2j+1) \end{cases}$$

These formulas are easily proved by means of (5.10).

The explicit results (5.13), (5.14), (5.15) also suggest that

$$(5.19)\qquad P_{j,s}(k) = 0 \qquad (j = 0, 1, \cdots, 2s-1).$$

This can be proved inductively using (5.10) in the form

$$(5.20)\qquad P_{j,s}(k) = P_{j+1,s}(k) - j(k-j+1)P_{j-1,s-1}(k).$$

Thus, to begin with,

$$P_{2s-1,s}(k) = P_{2s,s}(k) - (2s-1)(k-2s+2)P_{2s-2,s-1}(k) = 0,$$

by (5.12). In the next place, taking $j = 2s-2$, we get

$$P_{2s-2,s}(k) = P_{2s-1,s}(k) - (2s-2)(k-2s+3)P_{2s-s,s-1}(k) = 0.$$

Continuing in this way, we get

$$P_{j,s}(k) = 0 \qquad (1 \leqslant j \leqslant 2s-1).$$

Finally, taking $j = 1$ and replacing s by $s+1$ in (5.10), we have

$$P_{2,s+1}(k) = P_{1,s+1}(k) + kP_{0,s}(k),$$

which gives $P_{0,s}(k) = 0$.

6. We now put

$$P_j(k,x) = \sum_{2s\le j} (-1)^s P_{j,s}(k)x^{j-2s}, \qquad P_0 = 1, \qquad P_1 = x, \tag{6.1}$$

and

$$F(z) = F(k,x,z) = \sum_{j=0}^{\infty} P_j(k,x)\frac{z^j}{j!}\ . \tag{6.2}$$

By (5.10),

$$P_{j+1}(k,x) = \sum_{2s\le j+1} (-1)^s P_{j,s}(k)x^{j-2s+1} + j(k-j+1)\sum_{2s\le j+1} (-1)^s P_{j-1,s-1}(k)x^{j-2s+1},$$

so that

$$P_{j+1}(k,x) = xP_j(k,x) - j(k-j+1)P_{j-1}(k,x). \tag{6.3}$$

It follows from (6.2) and (6.3) that

$$F'(z) = \sum_{j=0}^{\infty} P_{j+1}(k,x)\frac{z^j}{j!} = x\sum_{j=0}^{\infty} P_j(k,x)\frac{z^j}{j!} - z\sum_{j=0}^{\infty}(k-j)P_j(k,x)\frac{z^j}{j!}$$
$$= xF(z) - kzF(z) + z^2F'(z).$$

Hence

$$\frac{F'(z)}{F(z)} = \frac{x-kz}{1-z^2}\ ,$$

which gives

$$F(k,x,z) = (1+z)^{\frac12(x+k)}(1-z)^{-\frac12(x-k)}. \tag{6.4}$$

It follows from the recurrence (6.3) that the polynomials

$$P_n(k,x) \qquad (n = 0,1,2,\cdots) \tag{6.5}$$

constitute a set of orthogonal polynomials in x. The polynomials have been discussed in [1, § 9]; in that paper the relationship with Euler numbers of higher order is stressed. If we put

$$(1+z)^{\frac12 x}(1-z)^{-\frac12 x} = \sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!},$$

so that $A_n(x) = P_n(0,x)$, then, by (6.4),

$$P_n(k,x) = \sum_{2s\le n} \frac{(-1)^s}{(n-2s)!}\binom{\frac12 k}{s} A_{n-2s}(x). \tag{6.6}$$

Returning to (5.9) and using (5.3), we have

$$j!\binom{k}{j} f_{j,k-j}(n) = \sum_{2s\le j} (-1)^s P_{j,s}(k)\cdot 2^{-k}\sum_{t=0}^{k}\binom{k}{t}(k-2t)^{n-2s}$$
$$= 2^{-k}\sum_{t=0}^{k}\binom{k}{t}(k-2t)^n\sum_{2s\le j}(-1)^s P_{j,s}(k)(k-2t)^{j-2s},$$

so that

$$(6.7)\qquad j!\binom{k}{j} f_{j,k-j}(n) = 2^{-k}\sum_{t=0}^{k}\binom{k}{t}(k-2t)^n P_j(k,k-2t) \qquad (0 \leqslant j \leqslant k).$$

We shall show that (6.7) holds for all j, that is, the right-hand side vanishes indentically for $j > k$. To prove this, consider the sum

$$2^{-k}\sum_{j=0}^{\infty}\frac{z^j}{j!}\sum_{n=0}^{\infty}\frac{y^n}{n!}\sum_{t=0}^{k}\binom{k}{t}(k-2t)^n P_j(k,k-2t) = 2^{-k}\sum_{t=0}^{k}\binom{k}{t}\sum_{n=0}^{\infty}\frac{y^n}{n!}(k-2t)^n\sum_{j=0}^{\infty}\frac{z^j}{j!}P_j(k,k-2t)$$

$$= 2^{-k}\sum_{t=0}^{k}\binom{k}{t}e^{(k-2t)y}(1+z)^{k-t}(1-z)^t = 2^{-k}e^{-ky}((1+z)e^{2y}+1-z)^k$$

$$= 2^{-k}((1+z)e^y+(1-z)e^{-y})^k = (\cosh y + z\sinh y)^k.$$

Since this is a polynomial of degree k in z, it follows that the right-hand side of (6.7) does indeed vanish for $j > k$ and all n. For example, for $k = 1$, we get

$$P_j(1,1) + (-1)^n P_j(1,-1) = 0 \qquad (j > 1).$$

Since this holds for all n, we have

$$(6.8)\qquad P_j(1,1) = P_j(1,-1) = 0 \qquad (j > 1).$$

Indeed, by (6.4),

$$\sum_{j=0}^{\infty} P_j(1,1)\frac{z^j}{j!} = 1+z, \qquad \sum_{j=0}^{\infty} P_j(1,-1)\frac{z^j}{j!} = 1-z,$$

in agreement with (6.8).

For $k = 2$ we get

$$2^n P_j(2,2) + 4\delta_{n,0}P_j(2,0) + (-2)^n P_j(2,-2) = 0 \qquad (j > 2).$$

This implies

$$P_j(2,2) = P_j(2,0) = P_j(2,-2) = 0 \qquad (j > 2).$$

Indeed, by (6.4),

$$\sum_{j=0}^{\infty} P_j(2,2)\frac{z^j}{j!} = (1+z)^2, \qquad \sum_{j=0}^{\infty} P_j(2,-2)\frac{z^j}{j!} = (1-z)^2, \qquad \sum_{j=0}^{\infty} P_j(2,0)\frac{z^j}{j!} = 1-z^2.$$

Since the determinant

$$\left|\binom{k}{t}(k-2t)^n\right| \neq 0 \qquad (t, n = 0, 1, \cdots, k),$$

the identical vanishing of the right-hand side of (6.7) implies

$$(6.9)\qquad P_j(k, k-2t) = 0 \qquad (j > k;\quad 0 \leqslant t \leqslant k).$$

This is indeed implied by (6.4), since

$$F(k, k-2t, z) = (1+z)^{k-t}(1-z)^t.$$

It follows from (6.9) and (6.1) that

$$(6.10)\qquad \sum_{2s\leqslant j}(-1)^s P_{j,s}(k)(k-2t)^{j-2s} = 0 \qquad (j > k;\ 0 \leqslant t \leqslant k);$$

it evidently suffices to take $t \leqslant k/2$. In particular, for $j = k+1$, (6.10) becomes

(6.11) $$\sum_{2s \leqslant j} (-1)^s P_{j,s}(j-1)(j-2r-1)^{j-2s} = 0 \qquad (2t < j)$$

For j even we consider

$$\sum_{s=0}^{j} (-1)^s P_{2j,s}(2j-1)(2j-2t-1)^{2j-2s} = 0 \qquad (0 \leqslant t < j).$$

Since $P_{j,0}(k) = 1$ this may be written in the form

(6.12) $$\sum_{s=1}^{j} (-1)^{s-1} P_{2j,s}(2j-1)(2r-1)^{2j-2s} = (2r-1)^{2j} \qquad (1 \leqslant r \leqslant j).$$

By Cramer's rule the system (6.12) has the solution

(6.13) $$P_{2j,s}(2j-1) = \frac{N_s}{D} \qquad (1 \leqslant s \leqslant j),$$

where

$$D = \det((2r-1)^{2s-2}) \qquad (r, s = 1, 2, \cdots, j)$$

and N_s is obtained from D by replacing the s^{th} column by $(2r-1)^{2t}$. Making use of a familiar theorem on the quotient of two alternants [4, Ch. 11], we get

(6.14) $$P_{2j,s}(2j-1) = c_s(1^2, 3^2, 5^2, \cdots, (2j-1)^2) \qquad (1 \leqslant s \leqslant j),$$

where $c_s(x_1, x_2, \cdots, x_j)$ denotes the s^{th} elementary symmetric function of the x_i.

For odd j in (6.11) we consider

$$\sum_{s=0}^{j} (-1)^s P_{2j+1,s}(2j)(2j-2t)^{2j-2s+1} = 0 \qquad (0 \leqslant t < j).$$

This may be written in the form

(6.15) $$\sum_{s=1}^{j} (-1)^{s-1} P_{2j+1,s}(2j)(2r)^{2j-2s+1} = (2r)^{2j+1} \qquad (1 \leqslant r \leqslant j).$$

Exactly as in the case of (6.12), the solution of the system (6.16) is given by

(6.16) $$P_{2j+1,s}(2j) = c_s(2^2, 4^2, 6^2, \cdots, (2j)^2) \qquad (1 \leqslant s \leqslant j).$$

where again c_s denotes the s^{th} symmetric function of the indicated arguments.

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