ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-272 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{j=0}^{m} \binom{r}{j}\binom{p}{m-j}\binom{q}{m-j}\binom{p+q-m+j}{j} \equiv C_m(p,q,r)$$

is symmetric in p, q, r.

H-273 Proposed by W. G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Consider, after Hoggatt and H-257, the array D, indicated below in which L_{2n+1} $(n = 0, 1, 2, \cdots)$ is written in staggered columns

1				
4	1			
11	4	1		
29	11	4	1	
76	29	11	4	1

i. Show that the row sums are $L_{2n+2} - 2$.
ii. Show that the rising diagonal sums are $F_{2n+3} - 1$, where L_{2n+1} is the largest element in the sum.
iii. Show that if the columns are multiplied by $1, 2, 3, \cdots$ sequentially to the right then the row sums are $L_{2n+3} - (2n+3)$.

SOLUTIONS

LOOK-SERIES

H-251 Proposed by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

Prove the identity:

$$\sum_{n=0}^{\infty} \frac{x^{n^2}}{[(x)_n]^2} = \sum_{n=0}^{\infty} \frac{x^n}{(x)_n},$$

where

$$(x)_n = (1-x)(1-x^2)\cdots(1-x^n), \qquad (x)_0 = 1.$$

Solution by the Proposer.

Define $f(z,y)$ by the following:

$$f(z,y) = \prod_{r=1}^{\infty} (1+y^{2r-1}z). \tag{1}$$

Then we may set

$$f(z,y) = \sum_{m=0}^{\infty} A_m(y)z^m ,$$

also observing that $f(0,y) = 1 = A_0(y)$.

Now,

$$f(y^2z,y) = (1+yz)^{-1}f(z,y),$$

which is readily derived from the definition of $f(z,y)$, i.e.,

$$f(z,y) = (1+yz)f(y^2z,y).$$

Translating this relation into series notation, we obtain the following:

$$\sum_{m=0}^{\infty} A_m(y)z^m = \sum_{m=0}^{\infty} A_m(y)y^{2m}z^m + \sum_{m=1}^{\infty} A_{m-1}(y)y^{2m-1}z^m .$$

This yields the simple recursion:

$$(1-y^{2m})A_m(y) = y^{2m-1}A_{m-1}(y),$$

with $A_0(y) = 1$. By an easy induction, we derive the formula:

$$A_m(y) = \frac{y^{m^2}}{(y^2)_m} \qquad (m = 0, 1, 2, \cdots).$$

Hence,

$$(2) \qquad f(z,y) = \prod_{r=1}^{\infty} (1+y^{2r-1}z) = \sum_{m=0}^{\infty} \frac{y^{m^2}}{(y^2)_m} z^m .$$

Similarly,

$$(3) \qquad f(z^{-1},y) = \prod_{r=1}^{\infty} (1+y^{2r-1}z^{-1}) = \sum_{n=0}^{\infty} \frac{y^{n^2}}{(y^2)_n} z^{-n} .$$

We now employ the well known Jacobi identity:

$$(4) \qquad f(z,y)\cdot f(z^{-1},y)\cdot \prod_{r=1}^{\infty} (1-y^{2r}) = \sum_{k=-\infty}^{\infty} y^{k^2}z^k .$$

Let $\theta(y)$ denote the coefficient of z^0 in $f(z,y)\cdot f(z^{-1},y)$. Multiplying the series in (2) and (3), we see that $\theta(y)$ is obtained by letting $m = n$; hence,

$$(5) \qquad \theta(y) = \sum_{k=0}^{\infty} \frac{y^{2k^2}}{\{(y^2)_k\}^2} .$$

However, from (4),

$$\theta(y) = \prod_{r=1}^{\infty} (1-y^{2r})^{-1} .$$

Making the substitution $x = y^2$ we obtain the result:

$$(6) \qquad \prod_{r=1}^{\infty} (1-x^r)^{-1} = \sum_{n=0}^{\infty} \frac{x^{n^2}}{\{(x)_n\}^2} .$$

Now the infinite product in (6) is the well known generating function for $p(n)$, the number of partitions of n; however, it is also equal to the series:

$$\sum_{n=0}^{\infty} \frac{x^n}{(x)_n} .$$

To establish this, define

$$g(z,x) = \prod_{r=1}^{\infty} (1-zx^r)^{-1},$$

and set

$$g(z,x) = \sum_{n=0}^{\infty} B_n(x)z^n,$$

observing that $g(0,x) = 1 = B_0(x)$. By inspection of the infinite product definition of $g(z,x)$, we may obtain the relation: $g(zx,x) = (1-zx)g(z,x)$; as before, translating this into the infinite series expansions, we obtain the recursion:

$$(1-x^n)B_n(x) = xB_{n-1}(x), \qquad B_0(x) = 1.$$

From this, we readily establish that

$$B_n(x) = x^n/(x)_n, \qquad n = 0, 1, 2, \cdots.$$

Hence, we have derived the following:

$$\text{(7)} \qquad \prod_{r=1}^{\infty} (1-x^r)^{-1} = g(1,x) = \sum_{n=0}^{\infty} p(n)x^n = \sum_{n=0}^{\infty} \frac{x^n}{(x)_n} = \sum_{n=0}^{\infty} \frac{x^{n^2}}{\{(x)_n\}^2},$$

for suitable region of convergence (actually, for $|x| < 1$.) This establishes the result.

Also solved by G. Lord and P. Tracy.

SUB PRODUCT

H-252 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let $A_{n\times n}$ be an $n \times n$ lower semi-matrix and $B_{n\times n}$, $C_{n\times n}$ be matrices such that $A_{n\times n}B_{n\times n} = C_{n\times n}$. Let $A_{k\times k}, B_{k\times k}, C_{k\times k}$ be the $k\times k$ upper left submatrices of $A_{n\times n}, B_{n\times n}$, and $C_{n\times n}$. Show $A_{k\times k}B_{k\times k} = C_{k\times k}$ for $k = 1, 2, \cdots, n$.

Solution by Paul S. Bruckman, University of Illinois at Chicago, Chicago Circle, Illinois.

Let a_{ij}, b_{ij} and c_{ij} denote the entries of A, B and C, respectively $(i,j = 1, 2, \cdots, n)$. By hypothesis,

$$\text{(1)} \qquad \sum_{r=1}^{n} a_{ir}b_{rj} = c_{ij}, \qquad i,j = 1, 2, \cdots, n;$$

$$\text{(2)} \qquad a_{ir} = 0 \text{ if } i < r.$$

Combining (1) and (2), we thus have:

$$\text{(3)} \qquad \sum_{r=1}^{i} a_{ir}b_{rj} = c_{ij}, \qquad i,j = 1, 2, \cdots, n.$$

If we impose the restriction: $i \leqslant k$, where $k \leqslant n$, then in view of (2) we may as well extend the sum in (3) as follows:

$$\text{(4)} \qquad \sum_{r=1}^{k} a_{ir}b_{rj} = c_{ij}, \qquad i = 1, 2, \cdots, k, \quad j = 1, 2, \cdots, n.$$

In particular,

$$\text{(5)} \qquad \sum_{r=1}^{k} a_{ir}b_{rj} = c_{ij}, \qquad i,j = 1, 2, \cdots, k.$$

This is equivalent to the desired result.

TRIPLE PLAY

H-253 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{t=0}^{k}\binom{(\beta-1)n+t+1}{t}\sum_{j=0}^{n-k-1}\binom{n-k-1}{j}\sum_{m=0}^{j}(-1)^{n+m+k+1}\binom{j}{m}$$

$$\times\sum_{r=0}^{n+m-t-j-1}\binom{j}{n+m-j-t-r-1}\binom{2j+r-1}{r} = 2^{n-k-1}\binom{\beta n}{k}\,.$$

where β is an arbitrary complex number and n and k are positive integers, $k < n$.

This identity, in the case $\beta = 2$, arose in solving a combinatorial problem in two different ways.

Solution by the Proposer.

To prove the identity we replace n by $n + k + 1$ and use

$$(1)\qquad \sum_{k=0}^{\infty}\binom{\alpha+\beta k}{k}w^k = \frac{x^{\alpha+1}}{(1-\beta)x+\beta}\,,$$

where $wx^{\beta} - x + 1 = 0$. This follows from the Lagrange expansion formula (cf. Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis,* I, Berlin, Springer, 1954, p. 125).

From (1) we have

$$(2)\qquad \sum_{k=0}^{\infty}2^n\binom{\beta n+\beta k+\beta}{k}w^k = \frac{2^n x^{\beta n+\beta+1}}{(1-\beta)x+\beta}\,,$$

where $wx^{\beta} - x + 1 = 0$. Also from (1) we get

$$\sum_{k=0}^{\infty}\sum_{t=0}^{k}\binom{(\beta-1)(n+k)+t+\beta}{t}w^k\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{j}(-1)^{n+m}\binom{j}{m}\sum_{r=0}^{n+k+m-t-j}\binom{j}{n+k+m-j-t-r}\binom{2j+r-1}{r}$$

$$= \sum_{k=0}^{\infty}w^k\sum_{t=0}^{\infty}\binom{(\beta-1)(n+k)+\beta t+\beta}{t}w^t\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{j}(-1)^{n+m}\binom{j}{m}$$

$$\times\sum_{r=0}^{n+k+m-j}\binom{j}{n+k+m-j-r}\binom{2j+r-1}{r} = \sum_{k=0}^{\infty}w^k\left(\frac{x^{(\beta-1)(n+k)+\beta+1}}{(1-\beta)x+\beta}\right)\sum_{j=0}^{n}\binom{n}{j}$$

$$\times\sum_{m=0}^{j}(-1)^{n+j-m}\binom{j}{m}\sum_{r=0}^{n+k-m}\binom{j}{n+k-m-r}\binom{2j+r-1}{r}\,,$$

where $wx^{\beta} - x + 1 = 0$.

Now

$$\frac{x^{\beta+1}}{(1-\beta)x+\beta}\sum_{k=0}^{\infty}w^k x^{(\beta-1)(n+k)}\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{j}(-1)^{n+j-m}\binom{j}{m}\sum_{r=0}^{n+k-m}\binom{j}{n+k-m-r}\binom{2j+r-1}{r}$$

$$= \frac{x^{\beta+1}w^{-n}}{(1-\beta)x+\beta}\sum_{j=0}^{n}\binom{n}{j}\sum_{r=0}^{\infty}\binom{2j+r-1}{r}\sum_{k=0}^{\infty}(x^{\beta-1}w)^{k+r}\sum_{m=0}^{\min\{k,j\}}(-1)^{n+j+m}\binom{j}{m}\binom{j}{k-m}$$

$$= \frac{x^{\beta+1}w^{-n}}{(1-\beta)x+\beta}\sum_{j=0}^{n}\binom{n}{j}(1-x^{\beta-1}w)^{-2j}\sum_{m=0}^{\infty}(-1)^{n+j+m}\binom{j}{m}\sum_{k=0}^{\infty}\binom{j}{k}(x^{\beta-1}w)^{k+m} =$$

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