A RESULT IN ANALYTIC NUMBER THEORY

K. JOSEPH DAVIS
Department of Mathematics, East Carolina University, Greenville, North Carolina 27834

The purpose of this note is to state and prove a result in analytic number theory that seems largely to have been overlooked. The usefulness of this result is illustrated by applying it to obtain an extremely simple proof of an estimate for a certain set of integers.

Let the letter p be used to denote primes.

Theorem 1. If f is multiplicative, then a necessary and sufficient condition that

$$\sum_{n=1}^{\infty} f(n)$$

converge absolutely is that

$$\prod_p \sum_{n=0}^{\infty} |f(p^n)|$$

converge. Furthermore, in the case of convergence,

$$\sum_{n=1}^{\infty} f(n) = \prod_p \left(\sum_{n=0}^{\infty} f(p^n) \right) .$$

Before we prove the theorem a few comments seem to be in order. The necessity is proved by Hardy and Wright [7, Theorem 286]. However, Hardy and Wright do not prove or even state the sufficiency condition above. Both necessary and sufficient conditions are stated by Ayoub [1, Theorem 1.5], but his statement of the sufficiency condition is careless and the proof given is not adequate.

Proof of Sufficiency. Let the increasing sequence of positive primes be denoted $p_1, p_2, \cdots$ and let t be a fixed integer. Then the general term in the product

$$\prod_{i=1}^{t} \left(\sum_{k=0}^{\infty} |f(p_i^k)| \right)$$

is of the form

$$|f(p_1^{\alpha_1})| |f(p_2^{\alpha_2})| \cdots |f(p_t^{\alpha_t})| = |f(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t})| ,$$

where

$$a_i \geqslant 0 \quad (1 \leqslant i \leqslant t).$$

The last equality is true because f is multiplicative. An integer n will appear in this product (as argument of f) if and only if it has no prime factors other than $p_1, p_2, \cdots, p_t$. By the unique factorization theorem it will then appear only once. Thus

$$\prod_{i \leqslant t} \sum_{k=0}^{\infty} |f(p_i^k)| = \sum_{(t)} |f(n)| ,$$

where the last summation is over all integers n whose only prime factors are in the set $p_1, p_2, \cdots, p_t$. Thus

$$\prod_p \sum_{k=0}^{\infty} |f(p^k)| = \lim_{t \to \infty} \prod_{i \leqslant t} \sum_{k=0}^{\infty} |f(p_i^k)| = \lim_{t \to \infty} \sum_{(t)} |f(n)| .$$

Now

$$A_t \equiv \sum_{n=1}^{p_t} |f(n)| \leq \sum_{(t)} |f(n)| \equiv B_t ,$$

since the summation on the right includes at least those on the left. Since $\{B_t\}$ converges, it is bounded, and therefore $\{A_t\}$ is a bounded, non-decreasing sequence. The fundamental theorem on monotone sequences applies and hence $\{A_t\}$ converges. But $\{A_t\}$ is a subsequence of the partial sums $\{s_n\}$of the series

$$\sum_{n=1}^{\infty} |f(n)| .$$

It follows that $\{s_n\}$ converges and the proof is complete.

Before we obtain the asymptotic result mentioned above we need the following definition. Let L represent the set of positive integers n with the property that p divides n implies that p^2 divides n. An integer in L is called a square-full integer. The characteristic function of L will be denoted by $1(n)$ and the summatory function of $1(n)$ will be denoted $L(x)$, so that

$$L(x) = \sum_{n \leq x} 1(n).$$

The proof of our result depends upon a famous theorem on series due to Kronecker (cf. [9, p. 129]). We give it in arithmetical form.

Lemma 1. If f is an arithmetical function and

$$\sum_{n=1}^{\infty} f(n)/n$$

is a convergent series, then f has mean value 0, that is,

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) = 0.$$

We now prove that L has density 0.

Theorem 2. The set L has density 0; that is,

$$\lim_{x \to \infty} \frac{L(x)}{x} = 0 .$$

Proof. By Lemma 1 we need only show that $\Sigma 1(n)/n$ converges. But by Theorem 1 and the multiplicativity of $1(n)$, it suffices to show that

$$\prod_p \left(\sum_{n=0}^{\infty} \frac{1(p^n)}{p^n} \right)$$

is convergent. By definition of $1(n)$

$$\prod_p \left(\sum_{n=0}^{\infty} \frac{1(p^n)}{p^n} \right) = \prod_p \left(1 + \frac{1(p)}{p} + \frac{1(p^2)}{p^2} + \dots \right) = \prod_p (1 + 1/p^2 + 1/p^3 + \dots) = \prod_p \left(1 + \frac{1}{p(p-1)} \right)$$

which is convergent.

Earlier proofs of this result were given by Feller and Tournier [6, §9] and Schoenberg [10, §12]. In addition Erdos and Szekeres [5], Hornfack [8], and Cohen [2], [3] have considered generalizations of the above problem. For a discussion of previous results including refinements of Theorem 2, see [3] and [4].

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ADDITIVE PARTITIONS I

V. E. HOGGATT, JR.
San Jose State University, San Jose, California 95192

David Silverman in July 1976 found the following property of the Fibonacci Numbers. This Theorem I was subsequently proved by Ron Evans, Harry L. Nelson, David Silverman, and Krishnaswami Alladi with myself, all independently.

Theorem I. The Fibonacci Numbers uniquely split the positive integers, N, into two sets A_0 and A_1 such that

$$A_0 \cup A_1 = N$$
$$A_0 \cap A_1 = \phi$$

and so that no two members of A_0 nor two members of A_1 add up to a Fibonacci number.

Theorem. (Hoggatt) Every positive integer $n \neq F_k$ is the sum of two members of A_0 or the sum of two members of A_1.

Theorem. (Hoggatt) Using the basic ideas above the Fibonacci Numbers uniquely split the Fibonacci Numbers, the Lucas Numbers uniquely split the Lucas Numbers and uniquely split the Fibonacci Numbers, and $\{5F\}_{n=2}^{\infty}$ uniquely splits the Lucas Sequence.

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