FIBONACCI NOTES
6. A GENERATING FUNCTION FOR HALSEY'S FIBONACCI FUNCTION

L. CARLITZ
Duke University, Durham, North Carolina 27706

1. Halsey [2] defined a Fibonacci function by means of

$$F_u = \sum_{k=0}^{m} \left\{ (u-x) \int_0^1 x^{u-2k-1}(1-x)^k dx \right\}, \tag{1.1}$$

where m is the unique integer satisfying

$$\tfrac{1}{2}u - 1 \leqslant m < \tfrac{1}{2}u, \tag{1.2}$$

that is

$$m = \begin{cases} [\tfrac{1}{2}u] & (\tfrac{1}{2}u \neq \text{integer}) \\ \tfrac{1}{2}u - 1 & (\tfrac{1}{2}u = \text{integer}) \end{cases}. \tag{1.3}$$

The definition (1.1) is equivalent to

$$F_u = \sum_{k=0}^{m} \binom{u-k-1}{k}, \tag{1.4}$$

where again m is defined by (1.3).

In a recent note [1], Bunder has proved that F_u as defined by (1.4) satisfies the recurrence

$$F_{u+1} - F_u - F_{u-1} = \begin{cases} 0 & (2m < u \leqslant 2m+1) \\ \binom{u-m-2}{m+1} & (2m+1 < u \leqslant 2m+2) \end{cases} \tag{1.5}$$

In the present note we construct a generating function for the sequence

$$\{F_{u+n}\} \qquad (j = 0, 1, 2, \cdots; 0 < u \leqslant 1).$$

We show that

$$\sum_{n=0}^{\infty} F_{u+n}x^n = \frac{(1-x^2)^{1-u}}{1-x-x^2} \qquad (0 < u \leqslant 1), \tag{1.6}$$

This result contains (1.5). It also follows from (1.6) that

$$F_{u+n} = \sum_{0 \leqslant 2j \leqslant n} \binom{u+j-2}{j} F_{n-2j+1} \qquad (0 < u \leqslant 1), \tag{1.7}$$

where the F_{n-2j+1} on the right are ordinary Fibonacci numbers. Thus F_{u+n} is a polynomial in u of degree $\leqslant n/2$. Indeed the coefficients of the polynomials

$$P_n(u) = n!\, F_{u+2n}, \qquad Q_n(u) = n!\, F_{u+2n+1} \tag{1.8}$$

are positive integers. For some properties of these coefficients see § 4 below.

2. Since m as defined by (1.2), is a function of u, we put $m = m(u)$. Then clearly

$$m(u+2j) = m(u) + j \qquad (j = 0, 1, 2, \cdots). \tag{2.1}$$

Assume that

Supported in part by NSF grant GP-37924X.

(2.2) $$0 < u \leqslant 2.$$

Then by (1.2), $m(u) = 0$ and $m(u+2j) = j$. Thus

$$F_{u+2j} = \sum_{k=0}^{j} \binom{u+2j-k-1}{k} = \sum_{k=0}^{j} \binom{u+j+k-1}{j-k} .$$

Hence

$$\sum_{j=0}^{\infty} F_{u+2j}x^{2j} = \sum_{j=0}^{\infty} x^{2j} \sum_{k=0}^{j} \binom{u+j+k-1}{j-k} = \sum_{k=0}^{\infty} x^{2k} \sum_{j=0}^{\infty} \binom{u+j+2k-1}{j} x^{2j}$$

$$= \sum_{k=0}^{\infty} x^{2k}(1-x^2)^{-u-2k} = \frac{(1-x^2)^{-u}}{1-x^2(1-x^2)^{-2}} ,$$

so that

(2.3) $$\sum_{j=0}^{\infty} F_{u+2j}x^{2j} = \frac{(1-x^2)^{2-u}}{(1-x^2)^2-x^2} \qquad (0 < u \leqslant 2).$$

Assume next that $0 < u \leqslant 1$, so that $m(u+1) = 0$ and $m(u+2j+1) = j$. Then as above

$$\sum_{j=0}^{\infty} F_{u+2j+1}x^{2j+1} = \sum_{j=0}^{\infty} x^{2j+1} \sum_{k=0}^{j} \binom{u+j+k}{j-k} = \sum_{k=0}^{\infty} x^{2k+1} \sum_{j=0}^{\infty} \binom{u+j+2k}{j} x^{2j}$$

$$= \sum_{k=0}^{\infty} x^{2k+1}(1-x^2)^{-u-2k-1} = \frac{x(1-x^2)^{-u-1}}{1-x^2(1-x^2)^{-2}} .$$

This gives

(2.4) $$\sum_{j=0}^{\infty} F_{u+2j+1}x^{2j+1} = \frac{x(1-x^2)^{1-u}}{(1-x^2)^2-x^2} \qquad (0 < u \leqslant 1).$$

Combining (2.3) and (2.4), we get

(2.5) $$\sum_{j=0}^{\infty} F_{u+j}x^{j} = \frac{(1-x^2)^{1-u}}{1-x-x^2} \qquad (0 < u \leqslant 1).$$

3. It is clear from (2.5), to begin with, that

(3.1) $$\lim_{u=1-0} F_{u+n} = F_{n+1} \qquad (n = 0, 1, 2, \cdots),$$

where F_{n+1} denotes an ordinary Fibonacci number. In the next place, writing (2.5) in the form

$$(1-x-x^2) \sum_{j=0}^{\infty} F_{u+j}x^{j} = \sum_{n=0}^{\infty} \binom{u+n-2}{n} x^{2n}$$

and equating coefficients, we get

(3.2) $$\begin{cases} F_{u+2n+1} - F_{u+2n} - F_{u+2n-1} = 0 \\ F_{u+2n+2} - F_{u+2n+1} - F_{u+2n} = \binom{u+2n}{2n+2} \end{cases} \qquad (0 < u < 1);$$

Since, by (1.2),

$$m(u+2n) = m(u+2n+1) = n \qquad (0 < u \leqslant 1),$$

and

$$2n < u+2n \leqslant 2n+1, \qquad 2n+1 < u+2n+1 \leqslant 2n+2,$$

it follows that (3.2) is equivalent to (1.5).

Since the right-hand side of (2.5) is equal to

$$\sum_{n=0}^{\infty} F_{n+1}x^n \sum_{j=0}^{\infty} \binom{u+j-2}{j} x^{2j},$$

we get

$$F_{u+n} = \sum_{0 \le 2j \le n} \binom{u+j-2}{j} F_{n-2j+1} \qquad (0 < u \le 1). \tag{3.3}$$

Alternatively, since

$$\frac{1-x^2}{1-x-x^2} = (1-x^2)\sum_{n=0}^{\infty} F_{n+1}x^n = 1+x+\sum_{n=2}^{\infty}(F_{n+1}-F_{n-1})x^n = 1+\sum_{n=1}^{\infty} F_n x^n,$$

it follows from (2.5) that

$$F_{u+n} = {\sum_{0 \le 2j \le n}}' \binom{u+j-1}{j} F_{n-2j} \qquad (0 < u \le 1), \tag{3.4}$$

where the dash indicates that, if F_0 occurs, it is to be taken equal to 1.

From (3.3) or (3.4) we infer that, for $0 < u \le 1$, F_{u+n} is a polynomial of degree $\le n/2$. Since

$$\binom{u+j-1}{j} = \frac{u(u+1)\cdots(u+j-1)}{j!},$$

the coefficients of the polynomial are positive. For example

$$F_u = 1, \quad F_{u+1} = 1, \quad F_{u+2} = 1+u, \quad F_{u+3} = 2+u, \quad F_{u+4} = \tfrac{1}{2}(6+3u+u^2),$$
$$F_{u+5} = \tfrac{1}{2}(10+5u+u^2), \quad F_{u+6} = \tfrac{1}{6}(48+23u+6u^2+u^3).$$

Another corollary of (3.3) may be noted. We have

$$F_{u+n+1}F_k - F_{u+n}F_{k+1} = \sum_{0 \le 2j \le n+1} \binom{u+j-2}{j} F_{n-2j+2}F_k$$
$$- \sum_{0 \le 2j \le n} \binom{u+j-2}{j} F_{n-2j+1}F_{k+1} = \sum_{0 \le 2j \le n+1} \binom{u+j-2}{j} (F_{n-2j+2}F_k - F_{n-2j+1}F_{k+1}).$$

Since

$$F_{m+1}F_n - F_mF_{n+1} = (-1)^{n+1}F_{m-n},$$

we get

$$F_{u+n+1}F_k - F_{u+n}F_{k+1} = (-1)^{k+1} \sum_{0 \le 2j \le n+1} \binom{u+j-2}{j} F_{n-k-2j+1}. \tag{3.5}$$

In particular, for $0 \le k \le n+1$, again making use of (3.3),

$$F_{u+n+1}F_k - F_{u+n}F_{k+1} \tag{3.6}$$
$$= (-1)^{k+1}F_{u+n+k} + (-1)^{k+1} \sum_{n-k+1 \le 2j \le n+1} \binom{u+j-2}{j} F_{2j-n+k-1} \qquad (0 \le k \le n).$$

For $k = n$ this reduces to

$$F_{u+n+1}F_n - F_{u+n}F_{n+1} = (-1)^{n+1}F_u + (-1)^{n+1} \sum_{0 < 2j \le n+1} \binom{u+j-2}{j} F_{2j-1}. \tag{3.7}$$

Similar results are implied by (3.4).

4. We have noted above that, for $0 < u \leqslant 1$, F_{u+n} is a polynomial of degree $\leqslant n/2$, indeed of degree $[n/2]$. Put

$$n!\,F_{u+2n} = P_n(u), \qquad n!\,F_{u+2n+1} = Q_n(u), \tag{4.1}$$

so that $P_n(u)$ and $Q_n(u)$ are of degree n. However we now think of them as defined for all u by means of (3.4) and (4.1). It follows from (1.5) that

$$\begin{cases} P_n(u) = nP_{n-1}(u) + nQ_{n-1}(u) + (u+n-2)\cdots u(n-1) \\ Q_n(u) = P_n(u) + nQ_{n-1}(u)\ . \end{cases} \tag{4.2}$$

Now put

$$P_n(u) = \sum_{k=0}^{n} p(n,k)u^k, \qquad Q_n(u) = \sum_{k=0}^{n} q(n,k)u^k\ .$$

We have also

$$(u+n-1)\cdots(u+1)u = \sum_{k=0}^{n} S(n,k)u^k,$$

where $S(n,k)$ denotes a Stirling number of the second kind. Thus

$$(u+n-1)\cdots u(u-1) = \sum_{k=0}^{n+1} (S(n,k-1) - S(n,k)u^k\ .$$

Hence (4.2) gives

$$q(n,k) = p(n,k) + nq(n-1),k) \tag{4.3}$$

and

$$p(n,k) = np(n-1,k) + nq(n-1,k) + S(n-1,k-1) - S(n-1,k)\,. \tag{4.4}$$

Using either (4.2) or (4.3) and (4.4), the following tables are easily computed.

$p(n,k)$:

$n \backslash k$	0	1	2	3	1
0	1				
1	1	1			
2	6	3	1		
3	48	23	6	1	
4	504	242	59	10	1

$q(n,k)$:

$n \backslash k$	0	1	2	3	4
0	1				
1	2	1			
2	10	5	1		
3	78	38	9	1	
4	816	394	95	14	1

It is evident from the recurrences (4.3) and (4.4) that the $p(n,k)$ and $q(n,k)$ are integers. Moreover, by (3.4), they are positive integers.

By (3.3) and (4.1),

$$P_n(1) = n!\,F_{2n+1}, \qquad Q_n(1) = n!\,F_{2n+2}\ . \tag{4.5}$$

This furnishes a partial check on the computed values. For example, using the table for $p(n,k)$, we get

$$\sum_{k=0}^{4} p(4,k) = 816 = 24.34 = 24F_9 .$$

Similarly

$$\sum_{k=0}^{4} q(4,k) = 1320 = 24.35 = 24F_{10} .$$

It is clear that

$$p(n,n) = q(n,n) = 1 \qquad (n = 0, 1, 2, \cdots). \tag{4.6}$$

Taking $k = n - 1$ in (4.3) and n in (4.4), we get

$$q(n, n-1) = p(n, n-1) + n \qquad (n \geqslant 1) \tag{4.7}$$

and

$$p(n+1, n) = 2(n+1) + S(n, n-1) - 1, \tag{4.8}$$

respectively. Since

$$S(n, n-1) = \tfrac{1}{2}n(n-1),$$

it follows that

$$\begin{cases} p(n, n-1) = \tfrac{1}{2}n(n+1) \\ q(n, n-1) = \tfrac{1}{2}n(n+3) = p(n+1, n) - 1 . \end{cases} \tag{4.9}$$

As for $k = 0$, it is evident from (3.4) that

$$\lim_{u=0} F_{u+n} = F_n ,$$

so that

$$p(n,0) = n!\,F_{2n}, \qquad q(n,0) = n!\,F_{2n+1} . \tag{4.10}$$

It would be of interest to find combinatorial interpretations of $p(n,k)$ and $q(n,k)$.

REFERENCES

1. M. W. Bunder, "On Halsey's Fibonacci Function," *The Fibonacci Quarterly,* Vol. 13, No. 2 (April 1975), pp. 209–210.
2. E. Halsey, "The Fibonacci Number F_u where u is not an Integer," *The Fibonacci Quarterly,* Vol. 3, No. 2 (April 1965), pp. 147–152.

****

[Continued from p. 245.]

(As a corollary, note that we have proved

$$F_{m+1}F_{m-1} - F_m^2 = \det(g^m) = (-1)^m .)$$

Then the lemma implies there is a sequence $\{m_j\}$ for which

$$g^{m_j} \to 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the p-adic topology. Thus we can choose $\{m_j\}$ so that $d(1, g^{m_j}) < p^{-j}$. Then p^j divides F_{m_j} and $1 - F_{m_j+1}$, which proves the theorem.

It is clear that one can vary G and g in the argument above to prove a class of theorems related to the well known one quoted.

****