LIMITING RATIOS OF CONVOLVED RECURSIVE SEQUENCES

V. E. HOGGATT, JR.
San Jose State University, San Jose, California 95192
and
KRISHNASWAMI ALLADI
Vivekananda College, Madras 600 004, India

It is a well known result that, for the Fibonacci numbers $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$,

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} .$$

See [1]. Our main result in this paper is that convolving linear recurrent sequences leaves limiting ratios unchanged. Some particular cases of our theorem prove an interesting study. It is indeed surprising that such striking limiting cases have been left unnoticed.

Definition 1. If $\{u_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers and if

$$\lambda = \lim_{n \to \infty} \frac{u_{n+1}}{u_n} ,$$

then λ is defined to be the limiting ratio of the sequence $\{u_n\}$.

Definition 2. If $\{u_n\}$ is a linear recurrence sequence

(1) $$a_0u_{n+r} + a_1u_{n+r-1} + a_2u_{n+r-2} + \cdots + a_ru_n = 0$$

then

$$a_0x^r + a_1x^{r-1} + \cdots + a_r = P_u(x)$$

is called the auxiliary polynomial for the sequence $\{u_n\}$.

Definition 3. If $\{u_n\} = U$ and $\{v_n\} = V$ are two linear recurrence sequences with generating functions

$$\frac{P(x)}{Q(x)} \quad \text{and} \quad \frac{R(x)}{S(x)},$$

respectively, we say $\{u_n\}$ and $\{v_n\}$ are *relatively prime* if

$$(P(x), S(x)) = (R(x), Q(x)) = 1.$$

The following classic result was known to Euler:

Lemma. If the auxiliary polynomial $P_u(x)$ for the sequence $\{u_n\}$ in (1) has a single root of largest absolute value, say λ, then

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lambda.$$

Let us call such a λ as a *dominant root* of $P_u(x)$. Moreover, let $\mathrm{Dom}(\alpha,\beta)$ represent the number with bigger absolute value.

The Lemma stated above leads to the following general theorem.

Theorem 1. Let

$$\{u_n\}_{n=0}^{\infty} \quad \text{and} \quad \{v_n\}_{n=0}^{\infty}$$

be two relatively prime linear recurrence sequences with auxiliary polynomials $P_u(x)$ and $P_v(x)$ whose dominant roots are λ_u, and λ_v. Then, if $\{w_n\}_{n=0}^{\infty}$ is the convolution sequence of $\{u_n\}$ and $\{v_n\}$,

(2) $$w_n = \sum_{k=0}^{n} v_k u_{n-k},$$

then

$$\lim_{n\to\infty} \frac{w_{n+1}}{w_n} = \mathrm{Dom}\,(\lambda_u, \lambda_v).$$

Proof. Consider a polynomial $P(x)$ with non-zero roots $a_1, a_2, \cdots, a_n$. Let $P^*(x)$ denote a polynomial with roots $1/a_1, 1/a_2, \cdots, 1/a_n$. We call $P^*(x)$ the *reciprocal* of $P(x)$. Now denote the reciprocals of $P_u(x)$ and $P_v(x)$ by $P_u^*(x)$ and $P_v^*(x)$, respectively. It is known from the theory of linear recurrence that

(3) $$\sum_{n=0}^{\infty} u_n x^n = \frac{R(x)}{P_u^*(x)}$$

and

(4) $$\sum_{n=0}^{\infty} v_n x^n = \frac{S(x)}{P_v^*(x)}$$

for some polynomials $R(x)$ and $S(x)$.

It is quite clear from (2), (3) and (4) that

(5) $$\sum_{n=0}^{\infty} w_n x^n = \frac{R(x)S(x)}{P_u^*(x)P_v^*(x)} = \frac{T(x)}{P_u^*(x)P_v^*(x)}$$

which reveals that $\{w_n\}$ is also a linear recurrence sequence. It is easy to prove that if $P_w(x)$ denotes the auxiliary polynomial of $\{w_n\}$, then its reciprocal $P_w^*(x)$ obeys

(6) $$P_w^*(x) = P_u^*(x)P_v^*(x).$$

It is clear that $1/\lambda_u$ and $1/\lambda_v$ are the roots of $P_u^*(x)$ and $P_v^*(x)$ with minimum absolute value, so that $\min\,(1/\lambda_u, 1/\lambda_v)$ is the root of $P_w^*(x)$ with minimum absolute value. But, since $P_w^*(x)$ is the reciprocal of $P_w(x)$, $\mathrm{Dom}\,(\lambda_u, \lambda_v)$ is the dominant root of $P_w(x)$. This together with the lemma proves

$$\lim_{n\to\infty} \frac{w_{n+1}}{w_n} = \lambda .$$

We state below some particular cases of the above theorem.

Theorem 2. Let $\{u_n\}_{n=0}^{\infty}$ be a linear recurrence sequence

$$u_{n+1} = u_n + u_{n-r}, \quad u_0 = 0, \quad u_1 = u_2 = u_3 = \cdots = u_r = 1, \quad r \in Z^+.$$

Let $g_{n,1}$ denote the first convolution sequence of $\{u_n\}_{n=0}^{\infty}$

$$u_n \sum_{n=0}^{\infty}$$

(7) $$g_{n,1} = \sum_{k=0}^{n} u_k u_{n-k}$$

and $g_{n,r}$ the r^{th} convolution $(u_n = g_{n,0})$

(8) $$g_{n,r} = \sum_{k=0}^{n} g_{k,r-1} u_{n-k}.$$

Then $\lim_{n\to\infty} u_{n+1}/u_n$ exists and

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{g_{n+1,r}}{g_{n,r}}$$

for every $r \in Z^+$.

Proof. The auxiliary polynomial for $\{u_n\}_{n=0}^{\infty}$ is $x^{r+1} - x^r - 1$. We will first prove that the root with largest absolute value is real. Denote the auxiliary polynomial by

$$P_u(x) = x^{r+1} - x^r - 1.$$

Clearly, $P_u(1) = -1 < 0$ and $P_u(\infty) = \infty$. Further,

$$\frac{dP_u(x)}{dx} = (r+1)x^r - rx^{r-1} > 0$$

for $1 < x < \infty$ so that $P_u(x) = 0$ for $1 \leqslant x < \infty$ at precisely one point, say λ_u. It is also clear that $P_u(x) > 0$ for $x > \lambda_u$ implies

(9) $$|x^{r+1}| > |x^r + 1|$$

for $x > \lambda_u$.

Let z_0 be a complex root of $P_u(x) = 0$ with $|z_0| > \lambda_u$. Now, since z_0 is a root of $P_u(x) = 0$,

$$|z_0^{r+1}| = |z_0^r + 1|.$$

But $|z_0| > \lambda_u$, and comparing with (9) we have

$$|z_0^{r+1}| \leqslant |z_0^r| + |1|,$$

a contradiction. One may also show similarly that there is no other root z_0 with $|z_0| = \lambda_u$ proving that λ_u is a dominant root of $P_u(x)$. This proves that the limiting ratio of $\{u_n\}$ exists and that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda_u.$$

Further, Theorem 1 gives

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{g_{n+1,r}}{g_{n,r}}$$

by induction on r and the definition of $g_{n,r}$ in (8).

Theorem 3. If $t,s \in Z^+$ and $t < s$, then

$$\lim_{n\to\infty} \frac{g_{n,t}}{g_{n,s}} = 0.$$

Proof. For the linear recurrence sequence $\{u_n\}$ satisfying

$$u_{n+1} = u_n + u_{n-r}, \quad u_0 = 0, \quad u_1 = u_2 = \cdots = u_r = 1,$$

define a companion sequence of polynomials

(10) $$u_{n+1}(x) = xu_n(x) + u_{n-r}(x)$$

$$u_0(x) = 0, \quad u_1(x) = 1, \quad u_2(x) = x, \cdots, u_r(x) = x^{r-1}.$$

Denote by $g_{n,0}(x) = u_n(x)$,

$$g_{n,1}(x) = \sum_{k=0}^{n} u_k(x)u_{n-k}(x),$$

and

(11) $$g_{n,t}(x) = \sum_{k=0}^{n} g_{k,r-1}(x)u_{n-k}(x).$$

One of us (K. A.) has established in [2] that

$$\frac{d^t u_n(x)}{t!\,dx^t} = g_{n,t}(x). \tag{12}$$

We know from (10) that

$$\frac{d^t u_{n+1}(x)}{dx^t} = x\,\frac{d^t u_n(x)}{dx^t} + t\cdot\frac{d^{t-1}u_n(x)}{dx^{t-1}} + \frac{d^t u_{n-r}(x)}{dx^t}. \tag{13}$$

Now, (12) makes (13) reduce to

$$g_{n+1,t}(x) = xg_{n,t}(x) + g_{n-r,t}(x) + g_{n,t-1}(x). \tag{14}$$

Note from (11) that $g_{n,t}(1) = g_{n,t}$ so that (14) can be rewritten as

$$g_{n+1,t} = g_{n,t} + g_{n-r,t} + g_{n,t-1}. \tag{15}$$

Dividing (15) throughout by $g_{n,t}$ we get

$$\frac{g_{n+1,t}}{g_{n,t}} = 1 + \frac{g_{n-r,t}}{g_{n,t}} + \frac{g_{n,t-1}}{g_{n,t}}. \tag{16}$$

We know from Theorem 2 that

$$\lim_{n\to\infty} g_{n+1,t}/g_{n,t} = \lambda_u \quad\text{and}\quad \lim_{n\to\infty} g_{n-r,t}/g_{n,t} = 1/\lambda_u^r,$$

so that (16) reduces to

$$\lambda_u = 1 + \frac{1}{\lambda_u^r} + \lim_{n\to\infty}\frac{g_{n,t-1}}{g_{n,t}}. \tag{17}$$

But, λ_u is the dominant root of $x^{r+1} - x^r - 1 = 0$ so that

$$\lim_{n\to\infty}\frac{g_{n,t-1}}{g_{n,t}} = 0.$$

This gives by induction

$$\lim_{n\to\infty}\frac{g_{n,t}}{g_{n,s}} = 0 \quad\text{for}\quad t < s,$$

proving Theorem 3.

Corollary. If $\{u_n\}$ is the Fibonacci sequence, then

$$\lim_{n\to\infty}\frac{g_{n+1,r}}{g_{n,r}} = \frac{1+\sqrt{5}}{2}$$

and

$$\lim_{n\to\infty}\frac{g_{n,t}}{g_{n,s}} = 0 \quad\text{for}\quad t < s.$$

We include the unproved theorem:

Theorem 4. If

$$g_{n+1,r}g_{n-1,r} - g_{n,r}^2 = w_n,$$

then

$$\lim_{n\to\infty}\frac{w_{n+1}}{w_n} = \lambda_u^2.$$

REFERENCES

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2. Krishnaswami Alladi, "On Polynomials Generated by Triangular Arrays," *The Fibonacci Quarterly,* Vol. 14, No. 4 (Dec. 1976), pp. 461–465.

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