PERIODIC CONTINUED FRACTION REPRESENTATIONS OF FIBONACCI-TYPE IRRATIONALS

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Consider the sequence $\{a_k\}_{k=1}^{\infty}$, where $a_k \geqslant 1 \ \forall\ k$, and also consider the sequence of *convergents*

(1) $$\frac{P_k}{Q_k} = [a_1, a_2, \cdots, a_k] \equiv a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_k}, \qquad k = 1, 2, \cdots.$$

It is known from continued fraction theory that $P_k = P_k(a_1, a_2, \cdots, a_k)$ and $Q_k = P_{k-1}(a_2, a_3, \cdots, a_k)$ are polynomial functions of the indicated arguments, with $Q_1 = 1$; moreover, the condition $a_k \geqslant 1 \ \forall\ k$ is sufficient to ensure that $\lim_{k \to \infty} P_k/Q_k$ exists. We call this limit the *value* of the infinite continued fraction $[a_1, a_2, a_3, \cdots]$; where no confusion is likely to arise, we will use the latter symbol to denote both the infinite continued fraction and its value. Clearly, this value is at least as great as unity, which is also true for all values of

$$P_k, \quad Q_k \quad \text{and} \quad \frac{P_k}{Q_k}, \quad k = 1, 2, \cdots.$$

The computation of the convergents of the infinite continued fraction $[a_1, a_2, a_3, \cdots]$ is facilitated by considering the matrix products

(2) $$\begin{pmatrix} P_k & P_{k-1} \\ Q_k & Q_{k-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}, \qquad k = 1, 2, \cdots,$$

where $P_0 \equiv 1$, $Q_0 \equiv 0$. Relation (2) is easily proved by induction, using the recursions

(3) $$P_{k+1} = a_{k+1}P_k + P_{k-1},$$

(4) $$Q_{k+1} = a_{k+1}Q_k + Q_{k-1}, \qquad k = 1, 2, \cdots.$$

Now, given a positive integer $n \geqslant 2$, suppose that we define the sequence $\{a_k\}_{k=1}^{\infty}$ as follows:

(5) $$a_1 = z, \quad a_2 = a_3 = \cdots = a_n = x, \quad a_{n+1} = 2z, \quad a_{k+n} = a_k, \qquad k = 2, 3, \cdots,$$

where $z \geqslant 1$, $x \geqslant 1$. Also, given that $n = 1$, we may define the sequence $\{a_k\}_{k=1}^{\infty}$ as follows:

(6) $$a_1 = z, \quad a_k = 2z, \qquad k = 2, 3, \cdots, \quad \text{where} \quad z \geqslant 1.$$

Let ϕ_n denote the value of the corresponding *periodic* infinite continued fraction; that is,

(7) $$\phi_n = [z; \overline{\underbrace{x, x, \cdots, x}_{n-1}, 2z}], \qquad n = 1, 2, \cdots.$$

Also, define θ_n as follows:

(8) $$\theta_n = z + \phi_n.$$

Thus, θ_n has a *purely periodic* continued fraction representation, namely

(9) $$\theta_n = [\overline{2z, \underbrace{x, x, \cdots, x}_{n-1}}].$$

We let P_k/Q_k denote the k^{th} convergent of the continued fraction given in (9) $(k = 1, 2, \cdots)$. In view of (2), note that

$$\begin{pmatrix} P_{n+1} & P_n \\ Q_{n+1} & Q_n \end{pmatrix} = \begin{pmatrix} 2z & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} 2z & 1 \\ 1 & 0 \end{pmatrix}.$$

Now, each matrix in the right member of the last expression is symmetric. Taking transposes of both sides leads to the result that the product matrix is itself symmetric, i.e.,

(10) $$P_n = Q_{n+1}.$$

We will return to this result later. Our concern is to evaluate θ_n, and thus ϕ_n, in terms of z, x and n. Another result which will be useful later is the special case of (4) with $k = n$, namely

(11) $$Q_{n+1} = 2zQ_n + Q_{n-1}.$$

Returning to (9), note that this is equivalent to the following:

(12) $$\theta_n = [2z, \underbrace{x, x, \cdots, x}_{n-1}, \theta_n].$$

This implies the equation

(13) $$\theta_n = \frac{\theta_n P_n + P_{n-1}}{\theta_n Q_n + Q_{n-1}}.$$

Clearing fractions in (13), we obtain a quadratic in θ_n, namely

(14) $$Q_n\theta_n^2 - (P_n - Q_{n-1})\theta_n - P_{n-1} = 0.$$

Rejecting the negative root of (14), we obtain the unique solution:

(15) $$\theta_n = \frac{P_n - Q_{n-1} + \sqrt{(P_n - Q_{n-1})^2 + 4P_{n-1}Q_n}}{2Q_n}.$$

Therefore, using (8), (11) and (10) in order, we obtain an expression for ϕ_n, which we shall find convenient to express in the form

(16) $$\phi_n = \sqrt{\frac{\left(\frac{P_n - Q_{n-1}}{2}\right)^2 \Big/ Q_n + P_{n-1}}{Q_n}}.$$

We will now show that (16) may be further simplified, and that depending on our choice of z, may be expressed in terms of a Fibonacci polynomial, with argument x. We digress for a brief review of these polynomials. The Fibonacci polynomials $F_m(x)$ are defined by the recursion:

(17) $$F_{m+2}(x) = xF_{m+1}(x) + F_m(x), \qquad m = 0, \pm 1, \pm 2, \cdots,$$

with initial values

(18) $$F_0(x) = 0, \qquad F_1(x) = 1.$$

The characteristic equation

(19) $$f^2 = xf + 1$$

has the two solutions

(20) $$\alpha(x) = \tfrac{1}{2}(x + \sqrt{x^2+4}), \qquad \beta(x) = \tfrac{1}{2}(x - \sqrt{x^2+4}),$$

which satisfy the relations

(21) $$\alpha(x)\beta(x) = -1, \quad \alpha(x) + \beta(x) = x, \quad \alpha(x) - \beta(x) = \sqrt{x^2+4}.$$

Closed form expressions for the F_m's are given by:

(22) $$F_m(x) = \frac{\alpha^m(x) - \beta^m(x)}{\alpha(x) - \beta(x)},$$

for all integers m. The Lucas polynomials are also defined by (17), but with initial values

(23) $$L_0(x) = 2, \qquad L_1(x) = x.$$

Closed forms for the Lucas polynomials $L_m(x)$ are given by:

(24) $$L_m(x) = \alpha^m(x) + \beta^m(x),$$

for all integers m. A convenient pair of formulas for extending the F_m's and L_m's to negative indices is the following.

(25) $$F_{-m}(x) = (-1)^{m-1}F_m(x),$$

(26) $$L_{-m}(x) = (-1)^m L_m(x), \qquad m = 0, 1, 2, \cdots.$$

Note that $F_m(1) = F_m$, $L_m(1) = L_m$, the familiar Fibonacci and Lucas *numbers,* respectively. The following additional relations may be verified by the reader:

(27) $$\alpha^r(x) = F_r(x)\cdot\alpha(x) + F_{r-1}(x);$$

(28) $$F_{m+2r}(x) = F_{r+1}^2(x)F_m(x) + 2F_{r+1}(x)F_r(x)F_{m-1}(x) + F_r^2(x)F_{m-2}(x);$$

(29) $$(x^2+4)F_{m+2r}(x) = L_{r+1}^2(x)F_m(x) + 2L_{r+1}(x)L_r(x)F_{m-1}(x) + L_r^2(x)F_{m-2}(x);$$

(30) $$\lim_{m\to\infty}\sqrt{\frac{F_{m+2r}(x)}{F_m(x)}} = \alpha^r(x), \quad \text{provided} \quad x > 0.$$

From (19),

$$\alpha^2(x) = x\alpha(x) + 1, \quad \text{or} \quad \alpha(x) = x + \frac{1}{\alpha(x)}.$$

Assuming $x \geqslant 1$, by iteration of the last expression, we ultimately obtain the purely periodic continued fraction expression for $\alpha(x)$, namely:

(31) $$\alpha(x) = [\bar{x}], \quad x \geqslant 1.$$

More generally, from (27),

$$\alpha^r(x)/F_r(x) = \alpha(x) + F_{r-1}(x)/F_r(x),$$

provided $F_r(x) \neq 0$. If, in particular, r is natural and $x \geqslant 1$, then in view of (31), we have:

$$\alpha^r(x)/F_r(x) = [\bar{x}] + F_{r-1}(x)/F_r(x) = [x + F_{r-1}(x)/F_r(x);\bar{x}] = [(xF_r(x)+F_{r-1}(x))/F_r(x);\bar{x}],$$

or, using (17) with $m = r-1$,

(32) $$\alpha^r(x)/F_r(x) = [F_{r+1}(x)/F_r(x);\bar{x}], \quad r \text{ natural}, \quad x \geqslant 1.$$

Comparing (30) and (32), it therefore seems reasonable to suppose that, for r natural and $x \geqslant 1$, the continued fraction expression for

$$\frac{1}{F_r(x)}\sqrt{\frac{F_{m+2r}(x)}{F_m(x)}}$$

should approximate, in some sense, the right member of (32). The exact relationship is both startling and elegant, and is our first main result. Before proceeding to it, however, we will develop a pair of useful lemmas.

Lemma 1. For all natural numbers r, let

(33) $$A_r(x) = \begin{pmatrix} F_{r+1}(x) & F_r(x) \\ F_r(x) & F_{r-1}(x) \end{pmatrix}.$$

Then

(34) $$A_r(x) = \{A_1(x)\}^r = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^r.$$

Proof. Let S be the set of natural numbers r for which (34) holds. Clearly, $1 \in S$. Suppose $r \in S$. Then, using the inductive hypothesis and (17), we obtain

$$\{A_1(x)\}^{r+1} = \{A_1(x)\}^r\cdot A_1(x) = \begin{pmatrix} F_{r+1}(x) & F_r(x) \\ F_r(x) & F_{r-1}(x) \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} xF_{r+1}(x) + F_r(x) & F_{r+1}(x) \\ xF_r(x) + F_{r-1}(x) & F_r(x) \end{pmatrix} = \begin{pmatrix} F_{r+2}(x) & F_{r+1}(x) \\ F_{r+1}(x) & F_r(x) \end{pmatrix} = A_{r+1}(x).$$

Hence, $r \in S \Rightarrow (r+1) \in S$.

By induction, Lemma 1 is proved.

Lemma 2. Suppose $[a_1, a_2, a_3, \cdots]$ converges. Then, for all $c > 0$,

$$c[a_1, a_2, a_3, \cdots] = \left[ca_1, \frac{a_2}{c}, ca_3, \frac{a_4}{c}, \cdots\right]. \tag{35}$$

Proof. Consider the convergents

$$\frac{P_k}{Q_k} = [a_1, a_2, a_3, \cdots, a_k], \qquad k = 1, 2, 3, \cdots.$$

Then

$$cP_k/Q_k = c\left\{a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_k}\right\} = ca_1 + \frac{c}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_k} = ca_1 + \frac{1}{(a_2/c)+}\,\frac{1/c}{a_3+}\cdots\frac{1}{a_k} = \cdots$$

$$= \left[ca_1, \frac{a_2}{c}, ca_3, \frac{a_4}{c}, \cdots, c^{(-1)^{k-1}}a_k\right].$$

Let

$$\phi = \lim_{k\to\infty} \frac{P_k}{Q_k} = [a_1, a_2, a_3, \cdots].$$

Then

$$\left[ca_1, \frac{a_2}{c}, ca_3, \cdots\right] = \lim_{k\to\infty}\left[ca_1, \frac{a_2}{c}, ca_3, \cdots, c^{(-1)^{k-1}}a_k\right] = \lim_{k\to\infty} cP_k/Q_k$$

$$= c\lim_{k\to\infty} P_k/Q_k = c\phi = c[a_1, a_2, a_3, \cdots]. \text{ Q.E.D.}$$

Before proceeding to the main theorems, we conclude the preliminary discussion with a brief table of $F_m(x)$ and $L_m(x)$, for ready reference:

m	$F_m(x)$	$L_m(x)$
0	0	2
1	1	x
2	x	x^2+2
3	x^2+1	x^3+3x
4	x^3+2x	x^4+4x^2+2
5	x^4+3x^2+1	x^5+5x^3+5x

Returning to (16), we may compute the required quantities from the matrix identity:

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} 2z & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^{n-1}.$$

However, using Lemma 1, this becomes:

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} 2z & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_n(x) & F_{n-1}(x) \\ F_{n-1}(x) & F_{n-2}(x) \end{pmatrix} = \begin{pmatrix} 2zF_n(x) + F_{n-1}(x) & 2zF_{n-1}(x) + F_{n-2}(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix}.$$

Substituting these quantities in (16), we thus obtain the result:

$$[z; \overline{\underbrace{x, x, \cdots, x}_{n-1}, 2z}] = \sqrt{\frac{z^2F_n(x) + 2zF_{n-1}(x) + F_{n-2}(x)}{F_n(x)}}, \tag{36}$$

for all natural n, provided $z \geqslant 1$, $x \geqslant 1$.

The following two theorems are easy consequences of (36):

Theorem 1. For all natural n and r, $x \geqslant 1$,

$$\frac{1}{F_r(x)}\sqrt{\frac{F_{n+2r}(x)}{F_n(x)}} = \left[\frac{F_{r+1}(x)}{F_r(x)}; \overline{\underbrace{x, x, \cdots, x}_{n-1}, \frac{2F_{r+1}(x)}{F_r(x)}}\right] \tag{37}$$

Proof: Let

$$z = \frac{F_{r+1}(x)}{F_r(x)}$$

in (36) and apply (28), with $m = n$. Since

$$z = x + \frac{F_{r-1}(x)}{F_r(x)} \geqslant x,$$

the condition $z \geqslant 1$ is clearly satisfied.

Theorem 2. For all natural n and r, $x \geqslant 1$,

(38) $$\frac{1}{L_r(x)}\sqrt{\frac{(x^2+4)F_{n+2r}(x)}{F_n(x)}} = \left[\frac{L_{r+1}(x)}{L_r(x)}; \overline{\underbrace{x, x, \cdots x}_{n-1}, \frac{2L_{r+1}(x)}{L_r(x)}}\right].$$

Proof. Let

$$z = \frac{L_{r+1}(x)}{L_r(x)}$$

in (36) and apply (29), with $m = n$. Since

$$z = x + \frac{L_{r-1}(x)}{L_r(x)} \geqslant x,$$

the condition $z \geqslant 1$ is clearly satisfied.

Corollary 1.

(39) $$\sqrt{\frac{F_{n+2}(x)}{F_n(x)}} = [x; \overline{\underbrace{x, x, \cdots, x}_{n-1}, 2x}] \quad ,$$

for all natural n, $x \geqslant 1$.

Proof. Set $r = 1$ in Theorem 1.

Corollary 2.

(40) $$\sqrt{\frac{F_{n+4}(x)}{F_n(x)}} = \begin{cases} [x^2+1; \overline{\underbrace{1, x^2, \cdots, 1, x^2}_{(\frac{1}{2}n-1)\text{ pairs}}, 1, 2x^2+2}], & n = 2, 4, 6, \cdots; \\ \left[x^2+1; \overline{\underbrace{1, x^2, \cdots, 1, x^2}_{\frac{1}{2}(n-1)\text{ pairs}}, \frac{2x^2+2}{x^2}, \underbrace{x^2, 1, \cdots, x^2, 1}_{\frac{1}{2}(n-1)\text{ pairs}}, 2x^2+2}\right], & n = 1, 3, 5, \cdots, x \geqslant 1. \end{cases}$$

Proof. Set $r = 2$ in Theorem 1. Then multiply both sides by $F_2(x) = x$, applying Lemma 2. Distinguishing between the cases n even and n odd leads to (40).

Corollary 3.

(41) $$\sqrt{\frac{F_{n+2}}{F_n}} = [1; \overline{\underbrace{1, 1, \cdots, 1}_{n-1}, 2}], \quad \text{for all natural } n,$$

Proof. Set $x = 1$ in Corollary 1.

Corollary 4.

(42) $$\sqrt{\frac{F_{n+4}}{F_n}} = [2; \overline{\underbrace{1, 1, \cdots, 1}_{n-1}, 4}], \quad \text{for all natural } n.$$

Proof. Set $x = 1$ in Corollary 2.

Corollary 5.

(43) $$\sqrt{\frac{(x^2+4)F_{n+2}(x)}{F_n(x)}} = \begin{cases} [x^2+2; \overline{\underbrace{1, x^2, \cdots, 1, x^2}_{(\frac{1}{2}n-1)\text{ pairs}}, 1, 2x^2+4}], & n = 2, 4, 6, \cdots; \\ \left[x^2+2; \overline{\underbrace{1, x^2, \cdots 1, x^2}_{\frac{1}{2}(n-1)\text{ pairs}}, \frac{2x^2+4}{x^2}, \underbrace{x^2, 1, \cdots, x^2, 1}_{\frac{1}{2}(n-1)\text{ pairs}}, 2x^2+4}\right], & n = 1, 3, 5, \cdots x \geqslant 1. \end{cases}$$

Proof. Set $r = 1$ in Theorem 2. Then multiply both sides by $L_1(x) = x$, applying Lemma 2. Distinguishing between the cases n even and n odd leads to (43).

Corollary 6.

$$\sqrt{\frac{5F_{n+2}}{F_n}} = [3; \overline{\underbrace{1, 1, \cdots, 1}_{n-1}, 6}], \quad \text{for all natural } n. \tag{44}$$

Proof. Set $x = 1$ in Corollary 5.

The continued fraction representations of corresponding expressions involving the *Lucas* polynomials are somewhat more complicated, since they contain fractions with numerators other than unity. The theory of such general continued fractions is more complex, and is not considered here. The interested reader may pursue this topic further, but will probably discover that the results found thereby will not be as elegant as those given in this paper.

The primary motivation for this paper came out of the diophantine equations studied in Bergum and Hoggatt [1].

REFERENCE

1. V. E. Hoggatt, Jr., and G. E. Bergum, "A Problem of Fermat and the Fibonacci Sequence," *The Fibonacci Quarterly,* to appear.

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PI-OH-MY!

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Though Π in circles may be found,
It's far from being a number round.
Not three, as thought in times Hebraic
(Indeed, this value's quite archaic!);
Not seven into twenty-two---
For engineers, this just won't do!
Three-three-three over one-oh-six
Is closer; but exactly? Nix!
The Hindus made a bigger stride
In valuing Π; if you divide
One-one-three into three-three-five.
This closer value you'll derive.
But Π's not even algebraic,
And so the previous lot are fake.
For those who deal in the abstract
Know it can never be exact
And are content to leave it go
Right next to omicron and rho.
As for the others, not as wise,
In circle-squarers' paradise,
They strain their every resource mental
To rationalize the transcendental!