Math. J., Vol. 32 (1965), pp. 437–446.
4. A.G. Shannon and A.F. Horadam, "Generating Functions for Powers of Third-Order Recurrence Sequences," *Duke Math. J.,* Vol. 38 (1971), pp. 791–794.
5. E. Lucas, *Theorie des Nombres,* Paris, 1891.

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A SET OF GENERALIZED FIBONACCI SEQUENCES SUCH THAT EACH NATURAL NUMBER BELONGS TO EXACTLY ONE

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1. INTRODUCTION

We shall prove there is an infinite array

1	2	3	5	8	.	.	.
4	6	10	16	26	.	.	.
7	11	18	29	47	.	.	.
9	15	24	39	63	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

in which every natural number occurs exactly once, such that past the second column every number in a given row is the sum of the two previous numbers in that row.

2. PROOF

Let α be the largest root of $z^2 - z - 1 = 0$, so $\alpha = (1+\sqrt{5})/2$. For every positive integer x let $f(x) = [\alpha x + \frac{1}{2}]$ where $[u]$ denotes the greatest integer in u. We require two lemmas: the first asserts that $f(x)$ is one-to-one, and the second asserts that the iterates of $f(x)$ form a sequence with the Fibonacci property.

Lemma 1. If x and y are positive integers and $x > y$ then $f(x) > f(y)$.

Proof. Since $\alpha(x-y) > 1$ we have $(\alpha x + \frac{1}{2}) - (\alpha y + \frac{1}{2}) > 1$, so $f(x) > f(y)$.

Lemma 2. If x and y are integers, and $y = [\alpha x + \frac{1}{2}]$, then $x + y = [\alpha y + \frac{1}{2}]$.

Proof. Write $\alpha x + \frac{1}{2} = y + r$, where $0 < r < 1$. Then

$$(1+\alpha)x + \frac{\alpha}{2} = \alpha y + \alpha r$$

so

$$x + y + r - \frac{1}{2} + \frac{\alpha}{2} = \alpha y + \alpha r \quad \text{and} \quad \alpha y + \frac{1}{2} = x + y + \frac{\alpha}{2} + (1-\alpha)r.$$

Since $1 < \alpha = 1.618 \cdots < 2$ we have $0 < \alpha - 1 < \frac{\alpha}{2} < 1$ and the result follows.

We now prove the theorem. Let the first row of the array consist of the Fibonacci numbers $1, 2 = f(1)$, $3 = f(2)$, $5 = f(3)$, $8 = f(5)$, and so on. The first positive integer not in this row is 4; let the second row be $4, 6 = f(4)$, $10 = f(6)$, $16 = f(10)$, and so on. The first positive integer not in the first or second row is 7; let the third row be $7, 11 = f(7)$, $18 = f(11)$, and so on. We see by Lemma 1 that there is no repetition. By Lemma 2 each row has the Fibonacci property. Finally, this process cannot terminate after a finite number of steps since the distances between successive elements in a row increase without bound. This completes the proof.

For the array just constructed, let a_n be the n^{th} number in the first column and b_n the n^{th} number in the second column. I conjecture that for $n \geq 2$ the difference $b_n - a_n$ is either a_i or b_i for some $i < n$.

We comment that the fact that $F_{n+1} = [\alpha F_n + \frac{1}{2}]$, where F_n is the n^{th} Fibonacci number, is Theorem III on p. 34 of the book *Fibonacci and Lucas Numbers*, Verner E. Hoggatt, Jr., Houghton Mifflin, Boston, 1969.

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