SOME SEQUENCE-TO-SEQUENCE TRANSFORMATIONS WHICH PRESERVE COMPLETENESS

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1. INTRODUCTION

A sequence $\{s_i\}_1^\infty$ of positive integers is termed *complete* if *every* positive integer N can be expressed as a distinct sum of terms from the sequence; it is well known ([1], Theorem 1) that if $\{s_i\}_1^\infty$ is nondecreasing with $s_1 = 1$, then a necessary and sufficient condition for completeness is

$$(1) \qquad s_{n+1} \leqslant 1 + \sum_1^n s_i \quad \text{for } n \geqslant 1.$$

Using this criterion for completeness, we will exhibit several transformations which convert a given complete sequence of positive integers into another sequence of positive integers without destroying completeness. Since the Fibonacci numbers $(F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 2)$ and the sequence of primes with unity adjoined $(P_1 = 1, P_2 = 2, 3, 5, 7, 11, 13, 17, \cdots)$ are examples of complete sequences, our results will yield as special cases some new complete sequences associated with the Fibonacci numbers and the primes.

2. QUANTIZED LOGARITHMIC TRANSFORMATION

Let $[x]$ denote the greatest integer contained in x, and define the function $<\cdot>$ by

$$<x> \; = \; 1 + [x] \quad \text{for all real } x.$$

Thus $<x>$ is the least integer $>x$ in contrast to $[x]$, the greatest integer $\leqslant x$. Both $<\cdot>$ and $[\cdot]$ may be thought of as quantizing characteristics in the sense that a non-integral x is rounded off to the integer immediately following x in the case of $<\cdot>$ or to the integer immediately preceding x when $[\cdot]$ is used. If x is an integer, then $[x] = x$ and $<x> = 1 + x$. The following lemma shows that $<\cdot>$ is subadditive:

Lemma 1. $$<x+y> \; \leqslant \; <x> + <y>.$$

Proof. If $x = [x] + \eta_x$ and $y = [y] + \eta_y$ with $0 \leqslant \eta_x, \eta_y < 1$, then

$$<x+y> \; = \; <[x] + [y] + \eta_x + \eta_y> \; \leqslant \; [x] + [y] + 2 = 1 + [x] + [y] + 1 = \; <x> + <y>.$$

Lemma 2. Let $\ln x$ denote the natural logarithm of x. Then for $x, y \geqslant 2$,

$$\ln(x+y) \leqslant \ln x + \ln y$$

that is, the logarithm is subadditive on the domain $[2, \infty)$.

Proof. For $x,y \geqslant 2$,

$$x + y \leqslant 2 \cdot \max(x,y) \leqslant \min(x,y) \max(x,y) = xy,$$

and $\ln(x+y) \leqslant \ln(xy) = \ln x + \ln y$, from the nondecreasing property of the logarithm.

Theorem 1. Let $\{s_i\}_1^\infty$ be a strictly increasing, complete sequence of positive integers. Then the sequence $\{<\ln s_i>\}_2^\infty$ is also complete.

Proof. By the assumed completeness,

$$s_{n+1} \leqslant 1 + \sum_1^n s_i \quad \text{for } n \geqslant 1.$$

Since $s_1 = 1$, we may write

$$s_{n+1} \leqslant 2 + \sum_2^n s_i \quad \text{for } n \geqslant 1;$$

hence,

$$\ln s_{n+1} \leqslant \ln\left(2 + \sum_2^n s_i\right),$$

and, on noting $s_i \geqslant 2$ for $i \geqslant 2$, it follows from Lemma 2 (by induction) that

$$\ln s_{n+1} \leqslant \ln 2 + \sum_2^n \ln s_i\,.$$

Now we may use the nondecreasing and subadditive (lemma 1) properties of $<\cdot>$ to conclude

$$<\ln s_{n+1}> \leqslant \left\langle \ln 2 + \sum_2^n \ln s_i \right\rangle \leqslant <\ln 2> + \sum_2^n <\ln s_i> = 1 + \sum_2^n <\ln s_i> \quad \text{for } n \geqslant 2.$$

Hence (noting $<\ln s_2> = <\ln 2> = 1$) by the completeness criterion, the sequence $\{<\ln s_i>\}_2^\infty$ is complete, proving the theorem.

The following theorem yields a similar conclusion for a class of functions ϕ where each ϕ possesses properties similar to that of the logarithmic function.

Theorem 2. Let $\{s_i\}_1^\infty$ be a nondecreasing complete sequence of positive integers and let $\phi(\cdot)$ be a function defined on the domain $x \geqslant 1$, nondecreasing and subadditive on that domain with $0 \leqslant \phi(1) < 1$. Then $\{<\phi(s_i)>\}_1^\infty$ is complete.

Proof. From

$$s_{n+1} \leqslant 1 + \sum_1^n s_i\,,$$

it follows that

$$\phi(s_{n+1}) \leqslant \phi\left(1 + \sum_1^n s_i\right) \leqslant \phi(1) + \sum_1^n \phi(s_i).$$

Then

$$<\phi(s_{n+1})> \leqslant <\phi(1)> + \sum_1^n <\phi(s_i)> = 1 + \sum_1^n <\phi(s_i)>,$$

so that, with $<\phi(1)> = 1$ and the completeness criterion, the sequence $\{\phi(s_i)\}_1^\infty$ is complete.

NOTE. Theorem 1 is not a special case of Theorem 2 since the logarithm is not subadditive on $[1, \infty)$. It is also clear that the domain of ϕ could be restricted to only those integers lying in $[1, \infty)$.

EXAMPLE. If $\phi(x) = \sqrt{x - 1/2}$ for $x \geqslant 1$, the reader may easily verify that ϕ is nondecreasing, subadditive and $0 \leqslant \phi(1) = \sqrt{1/2} < 1$. Therefore $\{<\sqrt{s_i - 1/2}>\}_1^\infty$ is complete whenever $\{s_i\}_1^\infty$ is a nondecreasing complete sequence of positive integers.

EXAMPLE. The function $\phi(x) = ax$ for $x \geqslant 1$ and some fixed $a > 0$ is nondecreasing and subadditive, and if

$0 < a < 1$, then $\phi(1) = a$ and ϕ satisfies the conditions of Theorem 2. Thus, for example, the sequence

$$\left\{\binom{s_i}{2}\right\}_1^\infty$$

is complete whenever $\{s_i\}_1^\infty$ is a nondecreasing complete sequence of positive integers.

EXAMPLE: If $P_1 = 1, P_2 = 2, 3, 5, 7, 11, \cdots$ denotes the sequence of primes (with unity adjoined); then it is well known [2] that $\{P_i\}_1^\infty$ is complete. Hence by Theorem 1, the sequence $\{\langle \ln P_i \rangle\}_2^\infty$ is also complete, and thus each positive integer N has an expansion of the form

$$N = \sum_2^\infty a_i \langle \ln P_i \rangle ,$$

where each a_i is binary (zero or one). The series is clearly finite, since $a_i = 0$ for $i \geqslant k$, where k is such that $\langle \ln P_k \rangle$ exceeds N.

It is of interest to prove the completeness of $\{\langle \ln P_i \rangle\}_2^\infty$ directly without using the completeness of $\{P_i\}_1^\infty$. In this manner, we avoid the implicit use of Bertrand's postulate which is normally invoked in showing the primes are complete.

Theorem 3. The sequence $\{\langle \ln P_i \rangle\}_2^\infty$ is complete.

Proof. Using Euler's classical argument, we observe that

$$1 + \prod_2^n P_i$$

is not divisible by $P_1, P_2, \cdots, P_n$ and therefore must have a prime divisor larger than P_n; that is

$$1 + \prod_2^n P_i \geqslant P_{n+1},$$

or

$$P_{n+1} \leqslant 1 + \prod_1^n P_i \leqslant 2 \prod_1^n P_i \quad \text{for} \quad n \geqslant 1.$$

Since the logarithm is an increasing function,

$$\ln P_{n+1} \leqslant \ln 2 + \sum_1^n \ln P_i$$

and consequently,

$$\langle \ln P_{n+1} \rangle \leqslant \langle \ln 2 \rangle + \sum_1^n \langle \ln P_i \rangle = 1 + \sum_1^n \langle \ln P_i \rangle$$

establishing the result by the completeness criterion.

3. LUCAS TRANSFORMATION

The transformation defined in the following theorem is called a Lucas Transformation since it corresponds to the manner in which the Lucas sequence is generated from the Fibonacci sequence.

Theorem 4. Let $\{u_i\}_1^\infty$ be a nondecreasing complete sequence with $u_1 = u_2 = 1$. Define a sequence $\{v_i\}_0^\infty$ by

$$\begin{cases} v_0 = 1 \\ v_1 = 2 \\ v_n = u_{n-1} + u_{n+1} \quad \text{for } n \geqslant 2. \end{cases}$$

Then $\{v_i\}_0^\infty$ is complete.

Proof. For $n \geqslant 1$,

$$v_{n+1} = u_n + u_{n+2} \leqslant 1 + \sum_{1}^{n-1} u_i + 1 + \sum_{1}^{n+1} u_i = (u_{n+1} + u_{n-1}) + (u_n + u_{n-2}) + \cdots + (u_3 + u_1) + u_2 + u_1 + 2$$

$$= v_n + v_{n-1} + \cdots + v_2 + u_2 + u_1 + 2 = v_n + v_{n-1} + \cdots + v_2 + v_1 + v_0 + 1 = 1 + \sum_{0}^{n} v_i ,$$

where we have used $u_2 + u_1 + 2 = 4 = v_1 + v_0 + 1$. Thus $v_0 = 1$ and

$$v_{n+1} \leqslant 1 + \sum_{0}^{n} v_i$$

for $n \geqslant 0$ which implies that $\{v_i\}_0^\infty$ is complete.

EXAMPLE: Let $u_i = F_i$, where $\{F_i\}_1^\infty$ is the Fibonacci sequence. Then the sequence defined by

$$v_0 = 1, \quad v_1 = 2, \quad v_n = F_{n-1} + F_{n+1} \quad \text{for } n \geqslant 2$$

is complete by Theorem 4. Moreover, recalling that the Lucas numbers $\{L_n\}_0^\infty$, defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+1} = L_n + L_{n-1} \quad \text{for } n \geqslant 1,$$

are also expressible by

$$L_n = F_{n-1} + F_{n+2} \quad \text{for } n \geqslant 2,$$

we see that $\{v_n\}_0^\infty$ is simply the sequence $\{L_n\}_0^\infty$ put in nondecreasing order by an interchange of L_0 and L_1. Completeness is not affected by a renumbering of the sequence; however, the inequality criterion for completeness must be applied only to nondecreasing sequences.

4. SUMMARY

If S denotes the set of all nondecreasing complete sequences of positive integers, we have considered certain transformations which map S into itself. In particular, it was shown, as special cases of the general results, that the sequences $\{\langle \ln F_n \rangle\}_3^\infty$, $\{\langle \ln P_n \rangle\}_2^\infty$ and $\{\langle \alpha F_n \rangle\}_2^\infty$ are complete sequences, where $\langle \cdot \rangle$ is defined by $\langle x \rangle = 1 + [x]$, $\{F_n\} = \{1, 1, 2, 3, 5, \cdots\}$ is the Fibonacci sequence, $\{P_n\} = \{1, 2, 3, 5, 7, 11, \cdots\}$ is the sequence of primes with unity adjoined and α is a fixed constant satisfying $0 < \alpha < 1$.

REFERENCES

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2. V.E. Hoggatt, Jr., and Bob Chow, "Some Theorems on Completeness," *The Fibonacci Quarterly,* Vol. 10, No. 5, 1972, pp. 551–554.

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