ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico 87131

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-370 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Solve the difference equation

$$u_{n+2} - 5u_{n+1} + 6u_n = F_n.$$

B-371 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let

$$S_n = \sum_{k=1}^{F_n} \sum_{j=1}^{k} T_j,$$

where T_j is the triangular number $j(j+1)/2$. Does each of $n \equiv 5 \pmod{15}$ and $n \equiv 10 \pmod{15}$ imply that $S_n \equiv 0 \pmod{10}$? Explain.

B-372 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let S_n be as in B-371. Does $S_n \equiv 0 \pmod{10}$ imply that n is congruent to either 5 or 10 modulo 15? Explain.

B-373 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California, and P. L. Mana, Albuquerque, New Mexico.

The sequence of Chebyshev polynomials is defined by

$$C_0(x) = 1, \quad C_1(x) = x, \quad \text{and} \quad C_n(x) = 2xC_{n-1}(x) - C_{n-2}(x)$$

for $n = 2, 3, \cdots$. Show that $\cos[\pi/(2n+1)]$ is a root of

$$[C_{n+1}(x) + C_n(x)]/(x+1) = 0$$

and use a particular case to show that $2\cos(\pi/5)$ is a root of

$$x^2 - x - 1 = 0.$$

B-374 Proposed by Frederick Stern, San Jose State University, San Jose, California.

Show both of the following:

$$F_n = \frac{2^{n+2}}{5}\,[(\cos(\pi/5))^n\cdot\sin(\pi/5)\cdot\sin(3\pi/5) + (\cos(3\pi/5))^n\cdot\sin(3\pi/5)\cdot\sin(9\pi/5)],$$

$$F_n = \frac{(-2)^{n+2}}{5}\,[(\cos(2\pi/5))^n\cdot\sin(2\pi/5)\cdot\sin(6\pi/5) + (\cos(4\pi/5))^n\cdot\sin(4\pi/5)\cdot\sin(12\pi/5)].$$

B-375 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Express

$$\frac{2^{n+1}}{5}\sum_{k=1}^{4}[(\cos(k\pi/5))^n\cdot\sin(k\pi/5)\cdot\sin(3k\pi/5)]$$

in terms of Fibonacci number, F_n.

SOLUTIONS

TRIANGULAR CONVOLUTION

B-346 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a closed form for

$$\sum_{k=1}^{n} F_{2k}T_{n-k} + T_n + 1,$$

where T_k is the triangular number

$$\binom{k+2}{2} = (k+2)(k+1)/2.$$

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

Using well-known generating functions one finds that

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} F_{2k}T_{n-k} + T_n + 1\right)x^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} F_{2k}T_{n-k}\right)x^n + \sum_{n=0}^{\infty} T_nx^n + \sum_{n=0}^{\infty} x^n$$

$$= \left(\sum_{n=0}^{\infty} F_{2n}x^n\right)\left(\sum_{n=0}^{\infty} T_nx^n\right) + \sum_{n=0}^{\infty} T_nx^n + \sum_{n=0}^{\infty} x^n$$

$$= \frac{x}{1-3x+x^2}\cdot\frac{1}{(1-x)^3} + \frac{1}{(1-x)^3} + \frac{1}{1-x}$$

$$= \frac{2-x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n+3}x^n.$$

Since for $k = 0$, $F_{2k}T_{n-k} = 0$, this implies that

$$\sum_{k=1}^{n} F_{2k}T_{n-k} + T_n + 1 = F_{2n+3}.$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Graham Lord, C. B. A. Peck, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, and the proposer.

A THIRD-ORDER ANALOGUE OF THE *F*'s

B-347 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let a, b, and c be the roots of $x^3 - x^2 - x - 1 = 0$. Show that

$$\frac{a^n - b^n}{a - b} + \frac{b^n - c^n}{b - c} + \frac{c^n - a^n}{c - a}$$

is an integer for $n = 0, 1, 2, \cdots$.

Solution by Graham Lord, Université Laval, Québec, Canada.

For $n = 0, 1, 2$ and 3 the expression, $E(n)$, above has the values 0, 3, 2 and 5, for all integers and demonstrating the recursion relation when

$$n = 0: E(n+3) = E(n+2) + E(n+1) + E(n).$$

This latter equation is readily proven since $a^3 = a^2 + a + 1$, etc. That $E(n)$ is an integer follows immediately, by induction, from this recursion relation.

Also solved by George Berzsenyi, Wray Brady, Clyde A. Bridger, Paul S. Bruckman, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, David Zeitlin, and the proposer.

PENTAGON RATIO

B-348 Proposed by Sidney Kravitz, Dover, New Jersey.

Let $P_1, \cdots, P_5$ be the vertices of a regular pentagon and let Q_1 be the intersection of segments $P_{i+1}P_{i+3}$ and $P_{i+2}P_{i+4}$ (subscripts taken modulo 5). Find the ratio of lengths Q_1Q_2/P_1P_2.

Solution by Charles W. Trigg, San Diego, California.

Extend P_4P_3 and P_4P_5 to meet P_1P_2 extended in A and B, respectively. Draw P_2P_5.

All diagonals of a regular pentagon of side e are equal, say, to d. Each diagonal is parallel to the side of the pentagon with which it has no common point. So, $AP_3P_5P_2$ is a rhombus. It follows that $AP_3 = AP_2 = d = BP_1 = BP_5$.

From similar triangles,

$$e/d = P_4P_3/P_3P_5 = P_4A/AB = (e+d)/(e+2d),$$

so, $d^2 - ed - e^2 = 0$ and $d = (\sqrt{5} + 1)e/2$.

Then,

$$Q_1Q_2/P_1P_2 = P_4Q_1/P_4P_2 = P_4P_3/P_4A = e/(e+d) = 2/(3+\sqrt{5}) = (3-\sqrt{5})/2 = 0.382 = \beta^2.$$

Furthermore,

$$Q_1Q_2/P_3P_5 = (Q_1Q_2/P_1P_2)(P_1P_2/P_3P_5) = (3-\sqrt{5})/(\sqrt{5}+1) = \sqrt{5} - 2 = 0.236 = -\frac{L_3 - F_3\sqrt{5}}{2} = -\beta^3.$$

Also solved by George Berzsenyi, Wray Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Dinh Thế Hùng, C. B. A. Peck, and the Proposer.

GENERATING TWINS

B-349 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let $a_0, a_1, a_2, \cdots$ be the sequence 1, 1, 2, 2, 3, 3, $\cdots$, i.e., let a_n be the greatest integer in $1 + (n/2)$. Give a recursion formula for a_n and express the generating function

$$\sum_{n=0}^{\infty} a_n x^n$$

as a quotient of polynomials.

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

Since the sequence of integers satisfies the relation $x_n = 2x_{n-1} - x_{n-2}$, the given sequence obviously satisfies the recursion formula $a_n = 2a_{n-2} - a_{n-4}$. The corresponding generating function is

$$\frac{x+1}{x^4 - 2x^2 + 1},$$

which may be proven by multiplying

$$\sum_{n=0}^{\infty} a_n x^n$$

by $x^4 - 2x^2 + 1$ and utilizing the above recurrence relation.

Also solved by Wray Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Graham Lord, David Zeitlin, and the Proposer.

CUBES AND TRIPLE SUMS OF SQUARES

B-350 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let a_n be as in B-349. Find a closed form for

$$\sum_{k=0}^{n} a_{n-k}(a_k + k)$$

in the case (a) in which n is even and the case (b) in which n is odd.

Solution by Graham Lord, Université Laval, Québec, Canada.

A closed form for the sum in case (a) is $(n+2)^3/8$, and in case (b) $(n+1)(n^2+5n+6)/8$. The proofs of these two are similar, only that of case (a) is given. With $n = 2m$,

$$\sum_{k=0}^{n} a_{n-k}(a_k+k) = \sum_{\ell=0}^{m} [1+m-\ell]\{[1+\ell]+2\ell\} + \sum_{\ell=0}^{m-1} [1+m-\ell-\tfrac{1}{2}]\{[1+\ell+\tfrac{1}{2}]+2\ell+1\}$$

$$= \sum_{0}^{m} (1+m-\ell)(1+3\ell) + \sum_{0}^{m-1} (m-\ell)(2+3\ell)$$

$$= (3m+1)(m+1) + 6m \sum_{0}^{m} \ell - 6 \sum_{0}^{m} \ell^2 = (m+1)^3.$$

Also solved by George Berzsenyi, Paul S. Bruckman, Herta T. Freitag, and the Proposer.

NON-FIBONACCI PRIMES

B-351 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $F_4 = 3$ is the only Fibonacci number that is a prime congruent to 3 modulo 4.

Solution by Graham Lord, Université Laval, Québec, Canada.

As $F_n \equiv 3 \pmod 4$ IFF $n = 6m + 4 = 2k$, then such an F_n factors $F_k L_k$, and so F_n is a prime IFF $F_k = 1$, that is IFF $n = 4$.

Also solved by Paul S. Bruckman, Michael Bruzinsky, Herta T. Freitag, Dinh Thê' Hũng, Bob Prielipp, Gordon Sinnamon, Lawrence Somer, and the Proposer.

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