# AN IDENTITY RELATING COMPOSITIONS AND PARTITIONS

**DEAN R. HICKERSON**
**University of California, Davis, California**

The following partition identity was proved in [1]:

*Theorem.* If $f(r,n)$ denotes the number of partitions of $n$ of the form $n = b_0 + b_1 + \cdots + b_s$, where for $0 \leqslant i \leqslant s-1$, $b_i \geqslant rb_{i+1}$, and $g(r,n)$ denotes the number of partitions of $n$, where each part is of the form $1 + r + r^2 + \cdots + r^i$ for some $i \geqslant 0$, then $f(r,n) = g(r,n)$.

In this paper, we will give a generalization of this theorem.

In [1], the parts of the partitions were listed in non-increasing order. It will, however, be more convenient for our purposes to list them in non-decreasing order.

The main result of this paper is given in the following theorem.

*Theorem 1.* Let $r_1, r_2, \cdots$ be integers. Let $c_0 = 1$ and, for $i \geqslant 1$, let $c_i = r_1c_{i-1} + r_2c_{i-2} + \cdots + r_ic_0$. Suppose that, for all $i \geqslant 0$, $c_i > 0$. For $i \geqslant 0$, let $t_i = c_0 + \cdots + c_i$ and define $T = \{t_0, t_1, t_2, \cdots\}$. Then, for $n \geqslant 0$, the number, $f(n)$, of compositions $b_0 + \cdots + b_s$ of $n$ in which $b_i \geqslant r_1b_{i-1} + r_2b_{i-2} + \cdots + r_ib_0$ for $1 \leqslant i \leqslant s$, is equal to the number, $g(n)$, of partitions of $n$ with parts in $T$.

*Proof.* Let $n = a_0t_0 + \cdots + a_st_s$ be a partition of $n$ counted by $g(n)$, where $a_s > 0$. Define, for $0 \leqslant i \leqslant s$,

$$b_i = \sum_{0 \leqslant j \leqslant i} a_{j+s-i}\, c_j \, .$$

Then

$$b_0 + \cdots + b_s = \sum_{0 \leqslant i \leqslant s} b_{s-i} = \sum_{0 \leqslant i \leqslant s}\ \sum_{0 \leqslant j \leqslant s-i} a_{i+j}c_j = \sum_{0 \leqslant k \leqslant s} \left( a_k \sum_{0 \leqslant j \leqslant k} c_j \right) = \sum_{0 \leqslant k \leqslant s} a_k t_k = n \, .$$

Also, for $0 \leqslant i \leqslant s$,

$$b_i = \sum_{0 \leqslant j \leqslant i-1} a_{j+s-i}c_j + a_sc_i > \sum_{0 \leqslant j \leqslant i-1} a_{j+s-i}c_j \geqslant 0 \, .$$

Therefore, $b_0 + \cdots + b_s$ is a composition of $n$. Moreover, for $1 \leqslant i \leqslant s$,

$$b_i \geqslant \sum_{1 \leqslant j \leqslant i} a_{j+s-i}c_j = \sum_{1 \leqslant j \leqslant i} \left( a_{j+s-i} \sum_{1 \leqslant k \leqslant j} r_kc_{j-k} \right) = \sum_{1 \leqslant k \leqslant i} \left( r_k \sum_{k \leqslant j \leqslant i} a_{j+s-i}c_{j-k} \right)$$

$$= \sum_{1 \leqslant k \leqslant i} \left( r_k \sum_{0 \leqslant j \leqslant i-k} a_{j+s-(i-k)}c_j \right) = \sum_{1 \leqslant k \leqslant i} r_kb_{i-k} \, .$$

Thus, $b_0 + \cdots + b_s$ is a composition of $n$ counted by $f(n)$.

This constitutes a mapping $\phi$ from the set of partitions counted by $g(n)$ into the set of compositions counted by $f(n)$. It suffices to show that $\phi$ is one-to-one and onto.

If $\phi$ is not one-to-one, then there exist distinct partitions $a_ot_o + \cdots + a_st_s$ and $a'_ot_o + \cdots + a'_{s'}t_{s'}$ of $n$ which yield the same composition. From the definition of $\phi$, it follows that $s = s'$. Let $i_o$ be the least $i \geqslant 0$ such that $a_{s-i} \neq a'_{s-i}$. Then

$$a_{s-i_0} = a_{s-i_0}c_0 = b_{i_0} - \sum_{1 \leqslant j \leqslant i_0} a_{s-(i_0-j)}c_j = b_{i_0} - \sum_{i \leqslant j \leqslant i_0} a'_{s-(i_0-j)}c_j = a'_{s-i_0}c_0 = a'_{s-i_0},$$

a contradiction. Hence $\phi$ is one-to-one.

We will now show that $\phi$ is onto. Let $b_0 + \cdots + b_s$ be a composition counted by $f(n)$. Define, for $0 \leqslant i \leqslant s$,

$$a_{s-i} = b_i - \sum_{1 \leqslant j \leqslant i} r_j b_{i-j}\ .$$

We claim that $a_0t_0 + \cdots + a_st_s$ is a partition counted by $g(n)$ whose image under $\phi$ is the composition $b_0 + \cdots + b_s$.

Clearly, $a_s = b_0 > 0$. Also, for $1 \leqslant i \leqslant s$,

$$b_i \geqslant r_1b_{i-1} + \cdots + r_ib_0 = \sum_{1 \leqslant j \leqslant i} r_jb_{i-j}$$

so $a_{s-i} \geqslant 0$. Also,

$$a_0t_0 + \cdots + a_st_s = \sum_{0 \leqslant i \leqslant s} a_{s-i}t_{s-i} = \sum_{0 \leqslant i \leqslant s}\left(b_i - \sum_{1 \leqslant j \leqslant i} r_jb_{i-j}\right)t_{s-i} = \sum_{0 \leqslant i \leqslant s} b_it_{s-i} - \sum_{0 \leqslant j < i \leqslant s} r_{i-j}b_jt_{s-i}$$

$$= \sum_{0 \leqslant j \leqslant s} b_jt_{s-j} - \sum_{0 \leqslant j \leqslant s}\left(b_j \sum_{j < i \leqslant s} r_{i-j}t_{s-i}\right) = \sum_{0 \leqslant j \leqslant s} b_j\left(t_{s-j} - \sum_{j < i \leqslant s} r_{i-j}t_{s-i}\right)$$

$$= \sum_{0 \leqslant j \leqslant s} b_{s-j}\left(t_j - \sum_{s-j < i \leqslant s} r_{i-s+j}t_{s-i}\right) = \sum_{0 \leqslant j \leqslant s} b_{s-j}\left(t_j - \sum_{1 \leqslant i \leqslant j} r_it_{j-i}\right)\ .$$

For $0 \leqslant j \leqslant s$, we have

$$t_j - \sum_{1 \leqslant i \leqslant j} r_it_{j-i} = \sum_{0 \leqslant k \leqslant j} c_k - \sum_{1 \leqslant i \leqslant j}\left(r_i \sum_{i \leqslant k \leqslant j} c_{k-i}\right)$$

$$= \sum_{0 \leqslant k \leqslant j}\left(c_k - \sum_{1 \leqslant i \leqslant k} r_ic_{k-i}\right) = c_0 + \sum_{1 \leqslant k \leqslant j}\left(c_k - \sum_{1 \leqslant i \leqslant k} r_ic_{k-i}\right).$$

By definition,

$$c_0 = 1 \quad \text{and} \quad c_k = \sum_{1 \leqslant i \leqslant k} r_ic_{k-i} \quad \text{for} \quad k \geqslant 1,$$

so

$$t_j - \sum_{1 \leqslant i \leqslant j} r_it_{j-i} = 1 \quad \text{and} \quad a_0t_0 + \cdots + a_st_s = \sum_{0 \leqslant j \leqslant s} b_{s-j} = n.$$

Therefore, $a_0t_0 + \cdots + a_st_s$ is a partition counted by $g(n)$.

We have

$$\sum_{0 \leqslant j \leqslant i} a_{j+s-i}c_j = \sum_{0 \leqslant k \leqslant i} a_{s-k}c_{i-k} = \sum_{0 \leqslant k \leqslant i} c_{i-k}\left(b_k - \sum_{1 \leqslant j \leqslant k} r_jb_{k-j}\right) = \sum_{0 \leqslant m \leqslant i} c_{i-m}b_m$$

$$- \sum_{\substack{0 \leqslant k \leqslant i\\ 0 \leqslant m \leqslant k}} c_{i-k}r_{k-m}b_m = b_i + \sum_{0 \leqslant m < i} b_m\left(c_{i-m} - \sum_{m < k \leqslant i} c_{i-k}r_{k-m}\right) = b_i + \sum_{0 \leqslant m < i} b_m\left(c_{i-m} - \sum_{1 \leqslant j \leqslant i-m} r_jc_{(i-m)-j}\right) = b_i$$

Therefore, the image under $\phi$ of the partition $a_0t_0 + \cdots + a_st_s$ is the composition $b_0 + \cdots + b_s$, so the proof is complete.

We will now determine when Theorem 1 is a partition identity. This occurs if and only if, for every $n \geq 0$, all compositions counted by $f(n)$ are partitions. Since $c_0 + c_1 + \cdots + c_i$ is a composition counted by $f(t_i)$, a necessary condition is that $c_0 \leq c_1 \leq c_2 \leq \cdots$. We now show that this condition is also sufficient.

*Theorem 2.* Suppose the hypotheses of Theorem 1 are satisfied, and, in addition, $c_0 \leq c_1 \leq c_2 \leq \cdots$. Then, for $n \geq 0$, the number of partitions $b_0 + \cdots + b_s$ of $n$ in which $b_i \geq r_1b_{i-1} + \cdots + r_ib_0$, for $1 \leq i \leq s$, is equal to the number of partitions of $n$ with parts in $T$.

*Proof.* It suffices to show that all compositions counted by $f(n)$ are partitions. Suppose $b_0 + \cdots + b_s$ is such a composition. Let $1 \leq k \leq s$. We will show, by induction on $i$, that, for $1 \leq i \leq k$,

$$b_k - b_{k-1} \geq (c_i - c_{i-1})b_{k-i} + \sum_{0 \leq j < k-i} b_j\left(r_{k-j} + \sum_{1 \leq \ell < i} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right).$$

Applying this with $i = k$ gives

$$b_k - b_{k-1} \geq (c_k - c_{k-1})b_0 \geq 0,$$

which will complete the proof.

We have

$$b_k - b_{k-1} \geq \sum_{0 \leq j < k} b_jr_{k-j} - b_{k-1} = (c_1 - c_0)b_{k-1} + \sum_{0 \leq j < k-1} b_jr_{k-j},$$

so the inequality holds for $i = 1$. Suppose it holds for $i = m - 1$, where $2 \leq m \leq k$. Then

$$\begin{aligned} b_k - b_{k-1} &\geq (c_{m-1} - c_{m-2})b_{k-m+1} + \sum_{0 \leq j < k-m+1} b_j\left(r_{k-j} + \sum_{1 \leq \ell < m-1} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right) \\ &\geq (c_{m-1} - c_{m-2})\left(\sum_{0 \leq j < k-m+1} b_jr_{k-j-m+1}\right) + \sum_{0 \leq j < k-m+1} b_j\left(r_{k-j} + \sum_{1 \leq \ell < m-1} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right) \\ &= \sum_{0 \leq j \leq k-m} b_j\left(r_{k-j} + \sum_{1 \leq \ell < m} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right) = b_{k-m}\left(r_m + \sum_{1 \leq \ell < m} (c_\ell - c_{\ell-1})r_{m-\ell}\right) \\ &\quad + \sum_{0 \leq j < k-m} b_j\left(r_{k-j} + \sum_{1 \leq \ell < m} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right). \end{aligned}$$

But

$$r_m + \sum_{1 \leq \ell < m} (c_\ell - c_{\ell-1})r_{m-\ell} = \sum_{0 \leq \ell < m} c_\ell r_{m-\ell} - \sum_{1 \leq \ell < m} c_{\ell-1}r_{m-\ell} = c_m - c_{m-1},$$

so

$$b_k - b_{k-1} \geq (c_m - c_{m-1})b_{k-m} + \sum_{0 \leq j < k-m} b_j\left(r_{k-j} + \sum_{1 \leq \ell < m} (c_\ell - c_{\ell-1})r_{k-j-\ell}\right),$$

and the inequality holds for $i = m$. This completes the induction and the proof.

The following is an important corollary of Theorem 2.

*Corollary.* Suppose $r_1, r_2, \cdots$ are non-negative integers with $r_1 \geq 1$. Define $T$ as above. Then, for $n \geq 0$, the number of partitions $b_0 + \cdots + b_s$ of $n$ in which $b_i \geq r_1b_{i-1} + \cdots + r_ib_0$, for $1 \leq i \leq s$, is equal to the number of partitions of $n$ with parts in $T$.

*Proof.* For $i \geq 1$, $c_i = r_1c_{i-1} + r_2c_{i-2} + \cdots + r_ic_0 \geq c_{i-1}$, and Theorem 2 applies.

We will now illustrate Theorems 1 and 2 and the corollary to Theorem 2 by some examples.

EXAMPLE 1. In the corollary, let $r_1 = r \geq 1$ and $r_2 = r_3 = \cdots = 0$. Then, for $i \geq 0$, $c_i = r^i$ and $t_i = 1 + r + \cdots + r^i$. Hence, for $n \geq 0$, the number of partitions $b_0 + \cdots + b_s$ of $n$ in which $b_i \geq rb_{i-1}$ for $1 \leq i \leq s$ is equal to the number of partitions of $n$ with parts of the form $1 + r + \cdots + r^i$ for $i \geq 0$. This is the result of [1].

EXAMPLE 2. In the corollary, let $r_1 = r_2 = 1$ and $r_3 = r_4 = \cdots = 0$. Then, for $i \geq 0$, $c_i = F_{i+1}$ and $t_i = F_{i+3} - 1$. Thus,

$$T = \{F_3 - 1, F_4 - 1, \cdots\} = \{1, 2, 4, 7, 12, \cdots\}.$$

For $n \geq 0$, the number of partitions of $n$ in which each part is greater than or equal to the sum of the two preceding parts is equal to the number of partitions of $n$ in which each part is 1 less than a Fibonacci number.

EXAMPLE 3. In the Corollary, let $r_1 = r_2 = \cdots = 1$. Then $c_0 = 1$ and, for $i \geq 1$, $c_i = 2^{i-1}$. Hence $t_i = 2^i$, for $i \geq 0$, and $T = \{1, 2, 4, 8, \cdots\}$. For $n \geq 0$, the number of partitions of $n$ in which each part is greater than or equal to the sum of all preceding parts is equal to the number of partitions of $n$ into powers of 2.

EXAMPLE 4. In Theorem 2, let $r_1 = -2$, $r_2 = -1$, $r_3 = r_4 = \cdots = 0$. Then, for $i \geq 0$, $c_i = i + 1$ and

$$t_i = \frac{(i+1)(i+2)}{2},$$

so $T = \{1, 3, 6, 10, 15, \cdots\}$. For $n \geq 0$, the number of partitions $b_0 + \cdots + b_s$ of $n$ in which $b_1 \geq 2b_0$ and, for $2 \leq i \leq s$, $b_i \geq 2b_{i-1} - b_{i-2}$ is equal to the number of partitions of $n$ into triangular numbers.

EXAMPLE 5. In Theorem 1, let $r_i = (-1)^{i+1}F_{i+2}$, for $i \geq 1$. Then $c_0 = 1$, $c_1 = 2$, $c_2 = c_3 = \cdots = 1$, so $t_0 = 1$ and $t_i = i + 2$ for $i \geq 1$. Hence, $T = \{1, 3, 4, 5, 6, \cdots\}$. For $n \geq 0$, the number of compositions $b_0 + \cdots + b_s$ of $n$ in which

$$b_i \geq 2b_{i-1} - 3b_{i-2} + 5b_{i-3} + \cdots + (-1)^{i+1}F_{i+2}b_0,$$

for $1 \leq i \leq s$, is equal to the number of partitions of $n$ with no part equal to 2.

## REFERENCE

1. Dean R. Hickerson, "A Partition Identity of the Euler Type," *Amer. Math. Monthly,* 81 (1974), pp. 627–629.

*******