[5] M. Kraitchik, *Le Problem du Cavalier* (Paris: Gauthiers-Villars et Cie, 1927).
[6] M. Kraitchik, *Mathematical Recreations* (New York: Dover, 1953).

*****

A FAMILY OF TRIDIAGONAL MATRICES

GERALD E. BERGUM
South Dakota State University, Brookings, SD 57007
and
VERNER E. HOGGATT, JR.
San Jose State University, San Jose, CA 95192

Consider the sequence of tridiagonal determinants $\left\{P_n^{(k)}(a, b, c)\right\}_{n=1}^{\infty}$ defined by $P_n^{(k)}(a, b, c) = P_n^{(k)} = \left|(a_{ij})\right|$ where

$$a_{ij} = \begin{cases} a, & i = j \\ b, & i = j - k \\ c, & i = j + k \\ 0, & \text{otherwise} \end{cases}.$$

We shall assume $P_n^{(1)} \neq 0$. The determinant $P_n^{(k)}$ has a's down the main diagonal, b's down the diagonal k positions to the right of the main diagonal and c's down the diagonal k positions below the main diagonal.

In [1], the authors discuss $\left\{P_n^{(2)}\right\}_{n=1}^{\infty}$ and find its generating function. This note deals with a relationship that exists between

$$\left\{P_n^{(k)}\right\}_{n=1}^{\infty} \text{ and } \left\{P_n^{(1)}\right\}_{n=1}^{\infty} \text{ for } k \geq 2.$$

The first few terms of $\left\{P_n^{(1)}\right\}_{n=}^{\infty}$ with $P_0^{(1)}$ defined as one are:

$$\begin{aligned}
P_0^{(1)} &= 1 \\
P_1^{(1)} &= a \\
P_2^{(1)} &= a^2 - bc \\
P_3^{(1)} &= a^3 - 2abc \\
P_4^{(1)} &= a^4 - 3a^2bc + b^2c^2 \\
P_5^{(1)} &= a^5 - 4a^3bc + 3ab^2c^2 \\
P_6^{(1)} &= a^6 - 5a^4bc + 6a^2b^2c^2 - b^3c^3 \\
P_7^{(1)} &= a^7 - 6a^5bc + 10a^3b^2c^2 - 4ab^3c^3
\end{aligned}$$

. . . .

By induction on n, it can be shown that

(A) $$P_{n+2}^{(1)} = aP_{n+1}^{(1)} - bcP_n^{(1)}, \; n \geq 1.$$

When $a = 1$ and $bc = -1$, we obtain the Fibonacci sequence. This result can also be found in [3] and [4].

The first few terms of $\left\{P_n^{(2)}\right\}_{n=1}^{\infty}$ can be found in [1] and are:

$$\begin{aligned}
P_1^{(2)} &= a = P_0^{(1)} P_1^{(1)} \\
P_2^{(2)} &= a^2 = \left[P_1^{(1)}\right]^2 \\
P_3^{(2)} &= a^3 - abc = P_1^{(1)} P_2^{(1)} \\
P_4^{(2)} &= (a^2 - bc)^2 = \left[P_2^{(1)}\right]^2 \\
P_5^{(2)} &= a^5 - 3a^2bc + 2ab^2c^2 = P_2^{(1)} P_3^{(1)} \\
P_6^{(2)} &= (a^3 - 2abc)^2 = \left[P_3^{(1)}\right]^2 \\
P_7^{(2)} &= a^7 - 5a^5bc + 7a^3b^2c^2 - 2ab^3c^3 = P_3^{(1)} P_4^{(1)} \\
P_8^{(2)} &= (a^4 - 3a^2bc + b^2c^2)^2 = \left[P_4^{(1)}\right]^2 \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

As with $\left\{P_n^{(1)}\right\}_{n=1}^{\infty}$, it can be shown by induction that

(B) $$P_n^{(2)} = aP_{n-1}^{(2)} - abcP_{n-3}^{(2)} + b^2c^2P_{n-4}^{(2)}, \; n \geq 5.$$

Not until our investigation of $P_n^{(3)}$ did we become suspicious of the fact that

(C) $$P_n^{(2)} = \begin{cases} P_q^{(1)} P_{q-1}^{(1)}, & n = 2q - 1 \\ \left[P_q^{(1)}\right]^2, & n = 2q \end{cases}.$$

The proof of the result (C) is as follows. Multiply the first and second rows of $P_n^{(2)}$ by $-c/a$ and add the results respectively to the third and fourth rows. Evaluate the new determinant using the first two columns to obtain

$$P_n^{(2)} = \begin{vmatrix} P_2^{(1)} & 0 & bP_1^{(1)} & 0 & 0 & 0 & \cdots \\ 0 & P_2^{(1)} & 0 & bP_1^{(1)} & 0 & 0 & \cdots \\ c & 0 & a & 0 & b & 0 & \cdots \\ 0 & c & 0 & a & 0 & b & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

Multiply the first and second rows of the new determinant by $-c/P_2^{(1)}$ and add the results respectively to the third and fourth rows. Evaluate the new determinant using the first two columns to obtain

$$P_n^{(2)} = \begin{vmatrix} P_3^{(1)} & 0 & bP_2^{(1)} & 0 & 0 & 0 & \cdots \\ 0 & P_3^{(1)} & 0 & bP_2^{(1)} & 0 & 0 & \cdots \\ c & 0 & a & 0 & b & 0 & \cdots \\ 0 & c & 0 & a & 0 & b & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

Repeating the process, using $-c/P_3^{(1)}$, we see that

$$P_n^{(2)} = \begin{vmatrix} P_4^{(1)} & 0 & bP_3^{(1)} & 0 & 0 & 0 & \cdots \\ 0 & P_4^{(1)} & 0 & bP_3^{(1)} & 0 & 0 & \cdots \\ c & 0 & a & 0 & b & 0 & \cdots \\ 0 & c & 0 & a & 0 & b & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

Let $n = 2q - 1$ and continue the technique above, evaluating by two columns at a time for $q - 1$ times until you obtain

$$P_n^{(2)} = \begin{vmatrix} P_{q-1}^{(1)} & 0 & bP_{q-2}^{(1)} \\ 0 & P_{q-1}^{(1)} & 0 \\ c & 0 & a \end{vmatrix} = P_q^{(1)} P_{q-1}^{(1)}.$$

If $n = 2q$ and we evaluate by two columns at a time for q times using the same technique as above we obtain

$$P_n^{(2)} = \begin{vmatrix} P_q^{(1)} & 0 \\ 0 & P_q^{(1)} \end{vmatrix} = \left[P_q^{(1)}\right]^2.$$

This procedure applied to $P_n^{(3)}$, where you evaluate by using three columns at a time instead of two, yields

$$P_n^{(3)} = \begin{cases} \left[P_{q-1}^{(1)}\right]^2 P_q^{(1)}, & n = 3q - 2 \\ P_{q-1}^{(1)}\left[P_q^{(1)}\right]^2, & n = 3q - 1 \\ \left[P_{q-1}^{(1)}\right]^3, & n = 3q \end{cases}.$$

In fact, it is easy to show if $n = kq - r$ that

(D) $$P_n^{(k)} = \left[P_{q-1}^{(1)}\right]^r \left[P_q^{(1)}\right]^{k-r} \text{ for } 0 \le r < k.$$

The authors found an alternate way of proving (C), but the technique did not apply if $k \geq 3$. This procedure is as follows. First show by induction, using (B) and (A), that

(E) $$P_{n+2}^{(2)} - bcP_n^{(2)} = P_{n+2}^{(1)}, \quad n \geq 1.$$

Next apply the results of Horadam [2], where

$$P_n^{(1)} = aP_{n-1}^{(1)} - bcP_{n-2}^{(1)}$$

is $W_n(a, b: p, q)$ with $a = 1$, $b = a$, $p = a$, and $q = bc$ to obtain

(F) $$aP_{2n-1}^{(1)} + b^2c^2\left[P_{n-2}^{(1)}\right]^2 = \left[P_n^{(1)}\right]^2.$$

Finally, using (A), (B), (E), (F), and induction, you can show (C).

REFERENCES

[1] V. E. Hoggatt, Jr., & G. E. Bergum, "A Combinatorial Problem Involving Recursive Sequences and Tridiagonal Matrices," *The Fibonacci Quarterly*, to appear.

[2] A. F. Horadam, "Basic Properties of a Certain Generalized Sequence of Numbers," *The Fibonacci Quarterly*, Vol. 3, No. 3 (Oct. 1965), pp. 161-176.

[3] *American Mathematical Monthly*, "Problem E-834," by Donald Walter, October 1948, p. 498.

[4] *American Mathematical Monthly*, "Solution to E-834," by Alex Tytun, June 1949, p. 409.