SOME POLYNOMIALS RELATED TO FIBONACCI AND EULERIAN NUMBERS

LEONARD CARLITZ
Duke University, Durham, North Carolina 27706

1. INTRODUCTION

Put

$$\frac{1}{1 - kx - x^2} = \sum_{n=0}^{\infty} c_{kn} x^n \tag{1.1}$$

and

$$C_n(y) = \sum_{k=0}^{\infty} c_{kn} y^k \qquad (n = 0, 1, 2, \ldots). \tag{1.2}$$

By (1.1),

$$\sum_{n=0}^{\infty} c_{kn} x^n = \sum_{j=0}^{\infty} x^j (k + x)^j = \sum_{j=0}^{\infty} \sum_{s=0} \binom{j}{s} k^{j-s} x^{j+s} = \sum_{n=0}^{\infty} x^n \sum_{2s \le n} \binom{n-s}{s} k^{n-2s},$$

so that

$$c_{kn} = \sum_{2s \le n} \binom{n-s}{s} k^{n-2s}. \tag{1.3}$$

Since c_{kn} is a polynomial in k of degree n, it follows that

$$C_n(y) = \frac{r_n(y)}{(1 - y)^{n+1}} \qquad (n = 0, 1, 2, \ldots), \tag{1.4}$$

where $r_n(y)$ is a polynomial in y of degree n. Moreover, since

$$c_{k,n+1} = kc_{k,n} + c_{k,n-1},$$

it follows from (1.2) that

$$C_{n+1}(x) = \sum_{k=0}^{\infty} (kc_{k,n} + c_{k,n-1}) x^k.$$

This gives

$$C_{n+1}(x) = C_n'(x) + C_{n-1}(x) \qquad (n \ge 1). \tag{1.5}$$

Hence, by (1.4),

Supported in part by NSF grant GP-37924X.

$$r_{n+1}(x) = (n+1)xr_n(x) + x(1-x)r_n'(x) + (1-x)^2 r_{n-1}(x) \qquad (n \geq 1) \tag{1.6}$$

with $r_0(x) = r_1(x) = 1$.

If we put

$$r_n(x) = \sum_{k=0}^{n} R_{n,k} x^k, \tag{1.7}$$

then, by (1.6), we get the recurrence

$$(n - k + 2)R_{n,k-1} + kR_{n,k} + R_{n-1,k} - 2R_{n-1,k-1} + R_{n-1,k-2}. \tag{1.8}$$

By means of (1.8) the following table is easily computed.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	•	1						
2	1	-1	2					
3	•	3	•	3				
4	1	•	14	4	5			
5	•	8	22	60	22	8		
6	1	6	99	244	279	78	13	
7	•	21	240	1251	2016	1251	240	21

It follows from (1.6) that

$$R_{n+1,n+1} = R_{n,n} + R_{n-1,n-1}.$$

Hence, since $R_{0,0} = R_{1,1} = 1$,

$$R_{n,n} = F_{n+1} \qquad (n = 0, 1, 2, \ldots). \tag{1.9}$$

Hoggatt and Bicknell [2] have conjectured that

$$R_{2n+1,k} = R_{2n+1,2n-k+2} \qquad (1 \leq k \leq 2n+1). \tag{1.10}$$

We shall prove that this is indeed true and that

$$R_{2n,2n-k+1} + (-1)^k \binom{2n+1}{k} = R_{2n,k} \qquad (1 \leq k \leq 2n). \tag{1.11}$$

The proof of (1.10) and (1.11) makes use of the relationship of $r_n(x)$ to the polynomial $A_n(x)$ defined by [1], [3, Ch. 2]

$$\frac{1-x}{1-xe^{(1-x)z}} = 1 + \sum_{n=1}^{\infty} A_n(x)\frac{x^n}{n!} \tag{1.12}$$

The relation in question is

$$r\ (x) = \sum_{2k \leq n} \binom{n-k}{k} (1 - x)^{2k} A_{n-2k}(x) \tag{1.13}$$

with $A_0(x) = 1$. The polynomial $A_n(x)$ is of degree n:

$$A_n(x) = \sum_{k=1}^{n} A_{n,k}\, x^k \qquad (n \geq 1), \tag{1.14}$$

where the $A_{n,k}$ are the Eulerian numbers. Since

$$A_n(x) = x^{n+1} A_n\left(\frac{1}{x}\right), \tag{1.15}$$

it is easily seen that (1.10) and (1.11) are implied by (1.13).

It seems difficult to find a simple explicit formula for $R_{n,k}$ or a simple generating function for $r_n(x)$. An explicit formula for $R_{n,k}$ is given in (2.11). As for a generating function, we show that

$$\sum_{n=0}^{\infty} r_n(x) \frac{z^n}{n!} = \sum_{n=0}^{\infty} A_n(x) f_n\left((1 - x)z\right)(1 - x)^{-n} z^n, \tag{1.16}$$

where

$$f_n(z) = \sum_{k=0}^{\infty} \frac{(k + n)!}{k!\,(2k + n)!} z^{2k+n}. \tag{1.17}$$

Moreover

$$f_n(z) = P_n(z) \cosh z + Q_n(z) \sinh z, \tag{1.18}$$

where $P_n(z)$, $Q_n(z)$ are polynomials of degree n, $n - 1$, respectively, that are given explicitly below.

While (1.16) is not a very satisfactory generating function, the explicit result (1.18) for $f_n(z)$ seems of some interest. It is reminiscent of the like result concerning Bessel functions of order half an integer [4, p. 52].

2. PROOF OF (1.10) AND (1.11)

By (1.2) and (1.3) we have

$$C_n(x) = \sum_{k=0}^{\infty} x^k \sum_{2s \leq n} \binom{n-s}{s} k^{n-2s} = \sum_{2s \leq n} \binom{n-s}{s} \sum_{k=0}^{\infty} k^{n-2s} x^k.$$

Since [3, p. 39]

$$\sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1 - x)^{n+1}},$$

it follows that

$$C_n(x) = \sum_{2s \leq n} \binom{n-s}{s} (1 - x)^{-n+2s} \quad A_{n-2s}(x)$$

and therefore

$$r_n(x) = \sum_{2s \le n} \binom{n-s}{s} (1-x)^{2s} A_{n-2s}(x). \tag{2.1}$$

Thus we have proved (1.13).

Note that by (1.7) and (1.14), (2.1) yields

$$R_{n,k} = \sum_{2s \le n} \sum_{j=0}^{k} (-1) \binom{2j}{j} \binom{n-s}{s} A_{n-zs,k-j}\,. \tag{2.2}$$

In the next place, since

$$A_n(x) = x^{n+1} A_n\left(\frac{1}{x}\right) \qquad (n > 0),$$

(2.1) gives

$$x^{n+1} r\left(\frac{1}{x}\right) = \sum_{2s \le n} \binom{n-s}{s} (1-x)^{2s} x^{n-2s+1} A_{n-2s}\left(\frac{1}{x}\right). \tag{2.3}$$

We now consider separately the cases n odd and n even.

Replacing n by $2n+1$, (2.3) becomes

$$x^{2n+2} r_{2n+1}\left(\frac{1}{x}\right) = \sum_{2s \le n} \binom{n-s}{s} (1-x)^{2s} A_{n-2s}(x),$$

so that

$$r_{2n+1}(x) = x^{2n+2} r_{2n+1}\left(\frac{1}{x}\right). \tag{2.4}$$

On the other hand

$$\begin{aligned} r^{2n+1} r_{2n}\left(\frac{1}{x}\right) &= \sum_{s=0}^{n-1} \binom{2n-s}{s} (1-x)^{2s} x^{2n-2s+1} A_{2n-2s}\left(\frac{1}{x}\right) + x(1-x)^{2n} \\ &= \sum_{s=0}^{n} \binom{2n-s}{s} (1-x)^{2s} A_{2n-2s}(x) - (1-x)^{2n+1}, \end{aligned}$$

so that

$$r_{2n}(x) = r^{2n+1} r_{2n}\left(\frac{1}{x}\right) + (1-x)^{2n+1}. \tag{2.5}$$

By (2.4) and (1.7) it follows at once that

$$R_{2n+1,k} = R_{2n+1,2n-k+2} \qquad (1 \le k \le 2n+1). \tag{2.6}$$

Similarly, by (2.5),

$$\sum_{k=0}^{2n} R_{2n,k} x^k = \sum_{k=0}^{2n} R_{2n,k} x^{2n-k+1} + \sum_{k=0}^{2n+1} (-1) \binom{2n+1}{k} x^k,$$

which gives

$$R_{2n,k} = R_{2n,2n-k+1} + (-1)^k \binom{2n+1}{k} \qquad (1 \le k \le 2n), \tag{2.7}$$

as well as

$$R_{2n,0} = 1 \qquad (n = 0, 1, 2, \ldots). \tag{2.8}$$

The companion formula

$$R_{2n+1,0} = 0 \qquad (n = 0, 1, 2, \ldots) \tag{2.9}$$

is implied by (2.4).

Clearly, by (1.9) and (2.6),

$$R_{2n+1,1} = F_{2n+2} \qquad (n = 0, 1, 2, \ldots) \tag{2.10}$$

while, by (2.7),

$$R_{2n,1} = F_{2n+1} - (2n+1) \qquad (n = 0, 1, 2, \ldots). \tag{2.11}$$

Since

$$A_n(y) = y \sum_{j=0}^{n} (y-1)^{n-j} \Delta^j 0^n,$$

where, as usual,

$$\Delta^j 0^n = \sum_{s=0}^{j} (-1)^{j-s} \binom{j}{s} s^n = j!S(n, j),$$

where $S(n, j)$ is a Stirling number of the second kind, (2.1) implies

$$\begin{aligned} r_n(x) &= x \sum_{2k \le n} \binom{n-s}{s} (1-x)^{2s} \sum_{j=0}^{n-2s} (x-1)^{n-2s-j} \Delta^j 0^{n-2s} \\ &= x \sum_{2s \le n} \binom{n-s}{s} \sum_{j=0}^{n-2s} (x-1)^{n-j} \Delta^j 0^{n-2s}. \end{aligned}$$

Hence

$$R_{n,k} = \sum_{2s \le n} \binom{n-s}{s} \sum_{j=0}^{n-2s} (-1)^{n\;\;j-k+1} \binom{n-j}{k-1} \Delta^j 0^{n-2s}. \tag{2.12}$$

For example, for $k = n$, (2.12) reduces to

$$R_{n,n} = \sum_{2s \le n} \binom{n-s}{s} = F_{n+1}.$$

3. GENERATING FUNCTIONS

To obtain a generating function for $r_n(x)$, we again make use of (2.1). Thus

$$\sum_{n=0}^{\infty} r_n(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{2k \le n} \binom{n-k}{k} (1 - x)^{2k} A_{n-2k}(x)$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{n+k}{k} \frac{z^{n+2k}}{(n+2k)!} (1 - x)^{2k} A_n(x).$$

If we put

$$f_n(z) = \sum_{k=0}^{\infty} \frac{(k+n)!}{k!(2k+n)!} z^{2k+n} = \sum_{k=0}^{\infty} \frac{(k+1)_n}{(2k+n)!} z^{2k+n}, \tag{3.1}$$

we get

$$\sum_{n=0}^{\infty} r_n(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty} A_n(x) f_n\big((1-x)z\big)(1-x)^{-n} z^n. \tag{3.2}$$

Clearly

$$f_0(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!},\quad 2f_1(z) = \sum_{k=0}^{\infty} \frac{2k+2}{(2k+1)!} z^{2k+1} = z\sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!},$$

so that

$$f_0(z) = \cosh z,\quad 2f_1(z) = z \cosh z + \sinh z. \tag{3.3}$$

For $n = 2$ we get

$$4f_2(z) = \sum_{k=0}^{\infty} \frac{4(k+1)(k+2)}{(2k+2)!} z^{2k+2} = \sum_{k=0}^{\infty} \frac{(2k+1)(2k+2) + 3(2k+2)}{(2k+2)!} z^{2k+2},$$

which reduces to

$$4f_2(z) = z^2 \cosh z + 3z \sinh z. \tag{3.4}$$

With a little more computation we find that

$$8f_3(z) = (z^3 + 3z^2) \cosh z + (6z^2 - 3) \sinh z. \tag{3.5}$$

These special results suggest that generally

$$2^n f_n(z) = P_n(z) \cosh z + Q_n(z) \sinh z, \tag{3.6}$$

where $P_n(z)$, $Q_n(z)$ are polynomials in z of degree n, $n-1$, respectively. We shall show that this is indeed the case and evaluate $P_n(z)$, $Q_n(z)$.

If we put

$$S_n(z) = P_n(z) + Q_n(z),\quad T_n(z) = P_n(z) - Q_n(z), \tag{3.7}$$

then (3.6) becomes

$$2^n f_n(z) = \frac{1}{2}\left(S_n(z)e^z + T_n(z)e^{-z}\right). \tag{3.8}$$

By (3.1) we have

$$2^n f_n(z) = \sum_{k=0}^{\infty} \frac{2^n(k+1)_n}{(2k+n)!} z^{2k+n}. \tag{3.9}$$

This suggests that we put

$$2^n(x+1)_n = \sum_{j=0}^{n} a_{nj}(2x+j+1)_{n-j}, \tag{3.10}$$

where the a_{nj} are independent of x. Clearly the a_{nj} are uniquely determined by (3.10). Indeed, rewriting (3.10) in the form

$$2^n\left(\frac{1}{2}(x-n)+1\right)_n = \sum_{j=0}^{n}(n-j)!\,a_{nj}\binom{x}{n-j},$$

it is evident, by finite differences, that

$$\begin{aligned} a_{n,n-j} &= \frac{2^n}{j!}\sum_{s=0}^{j}(-1)^{j-s}\binom{j}{s}\left(\frac{1}{2}(s-n)+1\right)_n \\ &= \frac{1}{j!}\sum_{s=0}^{j}(-1)^{j-s}\binom{j}{s}(s+n)(s+n-2)\cdots(s-n+2). \end{aligned} \tag{3.11}$$

Substituting from (3.10) in (3.9) we get

$$\begin{aligned} 2^n f_n(z) &= \sum_{k=0}^{\infty}\frac{z^{2k+n}}{(2k+n)!}\sum_{j=0} a_{nj}(2k+j+1) \\ &= \sum_{j=0}^{n} a_{nj} z^{n-j}\sum_{k=0}^{\infty}\frac{z^{2k+j}}{(2k+j)!} \\ &= \sum_{2j\le n} a_{n,2j} z^{n-2j}\left\{\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!} - \sum_{t=0}^{j-1}\frac{z^{2t}}{(2t)!}\right\} \\ &\qquad + \sum_{2j<n} a_{n,2j+1} z^{n-2j-1}\left\{\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!} - \sum_{t=0}^{j-1}\frac{z^{2t+1}}{(2k+1)!}\right\} \\ &= \sum_{2j\le n} a_{n,2j} z^{n-2j}\cosh z + \sum_{2j<n} a_{n,2j+1} z^{n-2j-1}\sinh z \\ &\quad -\sum_{2j\le n}\sum_{t<j} a_{n,2j}\frac{z^{n-2j-2t}}{(2t)!} - \sum_{2j<n}\sum_{t<j} a_{n,2j+1}\frac{z^{n-2j+2t}}{(2t+1)!}. \end{aligned} \tag{3.12}$$

Now

$$\sum_{2j \le n} \sum_{t<j} a_{n,2j} \frac{z^{n-2j+2t}}{(2t)!} + \sum_{2j<n} \sum_{t<j} a_{n,2j+1} \frac{z^{n-2j+2t}}{(2\quad + 1)!}$$

$$= \sum_{<2t<n} z^{n-2t} \left\{ \sum_{2t \le j < n} \frac{a_{n,2j}}{(2j - 2t)!} + \frac{a_{n,2j+1}}{(2j - 2t + 1)!} \right\}$$

$$= \sum_{0<2t<n} z^{n-2t} \sum_{2t \le j \le n} \frac{a_{n,j}}{(j - 2t)!}.$$

By (3.11)

$$\sum_{2t \le j \le n} \frac{a_{nj}}{(j - 2t)!}$$

$$= \sum_{2t \le j \le n} \frac{1}{(j - 2t)!(n - j)!} \sum_{s=0}^{n-j} (-1)^{n-j-s} \binom{n-j}{s} (s+n)(s+n-2) \cdots (s-n+2)$$

$$= \sum_{j=0}^{n-2t} \frac{1}{j!(n - 2t - j)!} \sum_{s=0}^{j} (-1)^{j-s} \binom{j}{s} (s+n)(s+n-2) \cdots (s-n+2)$$

$$= \frac{1}{(n-2t)!} \sum_{s=0}^{n-2t} \binom{n-2t}{s} (s+n)(s+n-2) \cdots (s-n+2) \sum_{j=s}^{n-2t} (-1)^{j-s} \binom{n-2t-s}{j-s}.$$

The inner sum vanishes unless $n = 2t + s$. Since $n > 2t$, the double sum must vanish. Therefore, (3.12) reduces to

$$2^n f_n(z) = \sum_{2j \le n} a_{n,2j} z^{n-2j} \cosh z + \sum_{2j<n} a_{n,2j+1} z^{n-2j-1} \sinh z. \tag{3.13}$$

Comparing (3.13) with (3.6), it is evident that

$$\begin{cases} P_n(z) = \sum\limits_{2j \le n} a_{n,2j} z^{n-2j} \\ Q_n(z) = \sum\limits_{2j<n} a_{n,2j+} \; z^{n-2j-1}. \end{cases} \tag{3.14}$$

Hence, as asserted above, $P_n(z)$, $Q_n(z)$ are polynomials of degree n, $n - 1$, respectively. It is in fact necessary to verify that $a_{n,0} \ne 0$, $a_{n,1} \ne 0$.

By (3.7) and (3.14) we have

$$S_n(z) = \sum_{j=0}^{n} a_{n,j} z^{n-j}, \quad T_n(z) = \sum_{j=0}^{n} (-1)\ a_{n,j} z^{n-j}. \tag{3.15}$$

4. ANOTHER EXPLICIT FORMULA

While we have found a_{nj} explicitly in (3.11), we shall now obtain another formula that exhibits a_{nj} as a polynomial in n of degree $2j$. To begin with we have, by (3.11),

$$\begin{aligned} e^z S_n(z) &= \sum_{k=0}^{\infty} z^k \sum_j \frac{j}{(k-j)!} a_{n,n-j} \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \sum_{s=0}^{j} (-1)^{j-s} \binom{j}{s} (s+n)(s+n-2) \cdots (s-n+2) \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{s=0}^{k} \binom{k}{s} (s+n)(s+n-2) \cdots (s-n+2) \sum_{j=s}^{k} (-1)^{j-s} \binom{k-s}{j-s}. \end{aligned}$$

It follows that

$$U_n(z) \equiv e^z S_n(z) = \sum_{k=0}^{\infty} \frac{(k+n)(k+n-2) \cdots (k-n+2)}{k!} z^k. \tag{4.1}$$

Then

$$zU_n'(z) - nU_n(z) = \sum_{k=0}^{\infty} \frac{(k+n)(k+n-2) \cdots (k-n)}{k!} z^k,$$

$$\left(zU_n'(z) - nU_n(z)\right) = \sum_{k=0}^{\infty} \frac{(k+n+1)(k+n-1) \cdots (k-n+1)}{k!} z^k = U_{n+1}(z).$$

Carrying out the differentiation this reduces to

$$S_{n+1}(z) = zS_n''(z) + (2z-n+1)S_n'(z) + (z-n+1)S_n(z). \tag{4.2}$$

Comparing coefficients we get

$$a_{n+1,j} = a_{nj} + (n-2j+3)a_{n,j-1} - (j-2)(n-j+2)a_{n,j-2}. \tag{4.3}$$

Hence, for $j = 0$, we get

$$a_{n,0} = 1. \tag{4.4}$$

For $j = 1$, (4.3) becomes

$$a_{n+1,1} = a_{n1} + (n+1)a_{n0},$$

which gives

$$a_{n1} = \binom{n+1}{2}. \tag{4.5}$$

For $n = 2$, (4.3) reduces to

$$a_{n+1,2} = a_{n2} + (n-1)a_{n1},$$

which gives

$$a_{n2} = 3\binom{n+1}{4}. \tag{4.6}$$

With a little more computation we get

$$a_{n3} = 15\binom{n+1}{6} - 3\binom{n+1}{4} \tag{4.7}$$

$$a_{n4} = 105\binom{n+1}{8} - 45\binom{n+1}{6} \tag{4.8}$$

$$a_{n5} = 3 \cdot 5 \cdot 7 \cdot 9\binom{n+1}{10} - 630\binom{n+1}{8} + 45\binom{n+1}{6}. \tag{4.9}$$

These special results suggest that generally

$$a_{n,j} = \sum_{2s<j} (-1)^s c_{js}\binom{n+1}{2j-2s}. \tag{4.10}$$

Indeed assuming that (4.10) holds up to j, it follows from (4.3) that

$$a_{n+1,j+1} - a_{n,j=1}$$

$$= (n-2j+1)\sum_{2s<j}(-1)^s c_{js}\binom{n+1}{2j-2s} - (j-1)(n-j+1)\sum_{2s<j-1}(-1)^s c_{j-1,s}\binom{n+1}{2j-2s-2}$$

$$= \sum_{2s<j}(-1)^s\binom{n+1}{2j-2s}\left\{(n-2j+1)c_{js} + (j-1)(n-j+1)c_{j-1,s-1}\right\}$$

$$= \sum_{2s\le j+1}(-1)^s c_{j+1,s}\binom{n+1}{2j-2s+1}$$

provided

$$(n-2j+2s+1)c_{j+1,s} = (2j-2s+1)\Big((n-2j+1)c_{js} + (j-1)(n-j+1)c_{j-1,s-1}\Big).$$

This gives

$$c_{j,s} = 2^{-j}\frac{(j-1)!\,(2j-2s)!}{s!\,(j-s)!\,(j-2s-1)!}. \tag{4.11}$$

Thus (4.10) becomes

$$a_{n,j} = \sum_{2s<j} 2^{-j} \frac{(j-1)!(2j-2s)!}{s!(j-s)!(j-2s-1)!} \binom{n+1}{2j-2s} \tag{4.12}$$

$a_{n,j}$:

n \ j	0	1	2	3	4	5
0	1					
1	1	1				
2	1	3				
3	1	6	3	-3		
4	1	10	15	-15		
5	1	15	45	-30	-45	+45

REFERENCES

[1] L. Carlitz, "Eulerian Numbers and Polynomials," *Mathematics Magazine*, Vol. 33 (1959), pp. 247-260.

[2] V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson, "Convolution Arrays for Jacobsthal and Fibonacci Polynomials," *The Fibonacci Quarterly* (to appear).

[3] John Riordan, *An Introduction to Combinatorial Analysis* (New York: John Wiley & Sons, 1958).

[4] G. N. Watson, *Theory of Bessel Functions*, 2d ed. (Cambridge: Cambridge Univ. Press, 1944.

*****