INTERPOLATION OF FOURIER TRANSFORMS ON SUMS OF FIBONACCI NUMBERS

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Our notation throughout this paper is that of [3]. Denote by $M(\mathbf{T})$ the Banach algebra of finite Borel measures on the circle group $\mathbf{T}$ and write $M_a(\mathbf{T})$ for those $\mu \in M(\mathbf{T})$ such that μ is absolutely continuous with respect to Lebesgue measure. Also $\mu \in M_d(\mathbf{T})$ if $\mu \in M(\mathbf{T})$ and μ is concentrated on a countable subset of $\mathbf{T}$.

The Fourier-Stieltjes transform $\hat{\mu}$ of the measure $\mu \in M(\mathbf{T})$ is defined by

$$\hat{\mu}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} d\mu(\theta) \qquad (n \in \mathbb{Z})$$

where $\mathbb{Z}$ is the additive group of integers. In this paper we prove that there is an infinite subset $\mathcal{A}$ of the set of Fibonacci numbers $\mathcal{F}$ such that

$$M_a(\mathbf{T})^\wedge|_{\mathcal{A}+\mathcal{A}} \subset M_d(\mathbf{T})^\wedge|_{\mathcal{A}+\mathcal{A}};$$

i.e., on $\mathcal{A}+\mathcal{A} = \{a + b : a,b \in \mathcal{A}\}$ any transform of an absolutely continuous measure can be interpolated by the transform of a discrete measure. To prove this, we shall need the following interesting result of S. Burr [1]:

A natural number m is said to be defective if the Fibonacci sequence $\mathcal{F} = \{f_n\}_1^\infty$ does not contain a complete system of residues modulo m.

Theorem 1: (Burr) A number m is not defective if and only if m has one of the following forms:

$$\begin{aligned}&5^k,\ 2\cdot 5^k,\ 4\cdot 5^k,\\ &3^j\cdot 5^k,\ 6\cdot 5^k,\\ &7\cdot 5^k,\ 14\cdot 5^k, \text{ where } k \geq 0,\ j \equiv 1.\end{aligned}$$

Let $\mathcal{S}^a$ denote the set of all integer accumulation points of $\mathcal{S} \subset \mathbb{Z}$ where the closure of $\mathcal{S}$ is taken with respect to the Bohr compactification $\overline{\mathbb{Z}}$ (see [3]) of $\mathbb{Z}$. In the sequel, we shall also need a theorem of Pigno and Saeki [6], which we now cite.

Theorem 2: The inclusion

$$M_a(\mathbf{T})^\wedge|_{\mathcal{S}} \subset M_d(\mathbf{T})^\wedge|_{\mathcal{S}}$$

obtains if and only if there is a measure $\mu \in M(\mathbf{T})$ such that $\hat{\mu}(\mathcal{S}) = 1$ and $\hat{\mu}(\mathcal{S}^a) = 0$.

We state and prove our main result:

Theorem 3: There is an increasing sequence $\mathcal{A} = \{f_n'\}_1^\infty$ of Fibonacci numbers such that

$$M_a(\mathbf{T})\big|_{\mathcal{A}+\mathcal{A}} \subset M_d(\mathbf{T})\big|_{\mathcal{A}+\mathcal{A}}.$$

Proof: By Theorem 1, we may find an increasing sequence $\mathcal{A} = \{f_n'\}_1^\infty$ of Fibonacci numbers such that

$$f_n' \equiv 5^n \pmod{2 \cdot 5^n} \text{ for all } n. \tag{1}$$

Now it follows from (1) that in the group of 5-adics (see [3, p. 107]) the only limit points of $\mathcal{A} + \mathcal{A}$ are 0 and each f_n'. Hence, to find the integer limit points of $\mathcal{A} + \mathcal{A}$ in $\overline{\mathbf{Z}}$ we need only look at 0 and each f_n'. Fix an f_n' and consider the arithmetic progression $\{2k + f_n' \, : \, k \in \mathbf{Z}\}$. This arithmetic progression is a neighborhood of f_n' in $\mathbf{Z}$ with the relative Bohr topology, and furthermore, $2k + f_n' = f_s' + f_t'$ is impossible because each member of $\mathcal{A}$ is odd [by (1)]. Thus, the only possible integer limit point of $\mathcal{A} + \mathcal{A}$ is 0.

Clearly the Dirac measure minus the Lebesgue measure separates $\mathcal{A} + \mathcal{A}$ and $\{0\}$ in the desired fashion. Hence we are done by Theorem 2.

Comments:

(i) Examples of related interpolation problems can be found in [2], [4], and [5].

(ii) It is an open question as to whether the sum set $\mathcal{F} + \mathcal{F}$ has the interpolation property of this paper. It is a result of the authors that if $\mathcal{A} = \{a^n : n \in \mathbf{Z}^+\}$, a any fixed positive integer, then $\mathcal{A} + \mathcal{A}$ has the interpolation property.

We wish to thank Professor V. E. Hoggatt, Jr., for the reference to [1].

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