GENERALIZED TRIBONACCI NUMBERS AND THEIR CONVERGENT SEQUENCES

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1. INTRODUCTION

In this note we consider sequences $\{a_n\}$ generated by the third-order recurrence relation

(1) $$a_{n+3} = \alpha a_{n+2} + \beta a_{n+1} + \gamma a_n, \quad n = 1, 2, 3, \ldots,$$

with real parameters α, β, γ and arbitrary real numbers a_1, a_2, a_3. Sequences like these have been considered by [3], [4], [5] with the TRIBONACCI sequence for $\alpha = \beta = \gamma = 1$ and $a_1 = a_2 = 1$, $a_3 = 2$ as a special case [1].

In this paper we give in the second section a general real representation for a_n using the roots of the auxiliary equation

(2) $$P_3(x): = x^3 - \alpha x^2 - \beta x - \gamma$$

in all possible cases. In the third section we characterize convergent sequences, give their limits and, finally, in the fourth section we consider various series with a_n as terms and give their limits by the use of a generating function.

2. REAL REPRESENTATION FOR $\{a_n\}$

According to the general theory of recurrence relations $\{a_n\}$ can be represented by

(3) $$a_n = Aq_1^{n-1} + Bq_2^{n-1} + Cq_3^{n-1}, \quad n = 1, 2, 3, \ldots,$$

where q_1, q_2, q_3 are the roots of the auxiliary equation $P_3(x) = 0$. The constants A, B, C are given by the linear equations system from (3) for $n = 1, 2, 3$ with prescribed "start" numbers a_1, a_2, a_3. The determinant of this system is the VANDERMONDE determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q_3 \\ q_1^2 & q_2^2 & q_3^2 \end{vmatrix} = \prod_{\substack{i,k=1 \\ i>k}} (q_i - q_k) = (q_2 - q_1)(q_3 - q_1)(q_3 - q_2)$$

which does not vanish for distinct q_1, q_2, q_3. In this case we get

$$A = \frac{q_2 q_3 a_1 - (q_2 + q_3)a_2 + a_3}{(q_3 - q_1)(q_2 - q_1)} \qquad B = \frac{-q_1 q_3 a_1 + (q_1 + q_3)a_2 - a_3}{(q_3 - q_2)(q_2 - q_1)}$$

$$C = \frac{q_1 q_2 a_1 - (q_1 + q_2)a_2 + a_3}{(q_3 - q_2)(q_3 - q_1)}$$

So we have by (3) a real representation for a_n, if the roots of $P_3(x)$ are distinct and real. If two roots are equal, e.g., $q_2 = q_3$, we get from (3) the limit as q_3 approaches q_2

(4) $$a_n = Dq_1^{n-1} + E_n q_2^{n-1}, \; n = 1, 2, 3, \ldots,$$

with

$$D = \frac{q_2^2 a_1 - 2q_2 a_2 + a_3}{(q_2 - q_1)^2},$$

$$E_n = \frac{1}{(q_2 - q_1)^2}\left\{[(n-3)q_2 - (n-2)q_1]a_1 q_1 - \left[(n-3)q_2 - (n-1)\frac{q_1^2}{q_2}\right]a_2 + \left[(n-2) - (n-1)\frac{q_1}{q_2}\right]a_3\right\}.$$

If all roots are equal, we get from (4) the limit as q_2 approaches q_1

(5) $$a_n = F_n q_1^{n-1}, \; u = 1, 2, 3, \ldots,$$

with

$$F_n = \frac{(n-2)(n-3)}{2}a_1 - \frac{(n-1)(n-3)}{q_1}a_2 + \frac{(n-1)(n-2)}{2q_1^2}a_3.$$

In the last of the possible cases for the roots of $P_3(x) = 0$ we have one real root q_1 and two conjugate complex ones q_2, q_3. Writing $q_2 = re^{i\varphi}$, $q_3 = \bar{q}_2 = re^{-i\varphi}$ we get

(6) $$a_n = Gq_1^{n-1} + H_n r^{n-1}, \; n = 1, 2, 3, \ldots,$$

with

$$G = \frac{r^2 a_1 - 2r\cos\varphi\, a_2 + a_3}{r^2 - 2rq_1\cos\varphi + q_1^2},$$

$$H_n = \frac{(a_1 q_1 - a_2)r\sin(n-3)\varphi + (a_3 - a_1 q_1^2)\sin(n-2)\varphi + (a_2 q_1 - a_3)\frac{q_1}{r}\sin(n-1)\varphi}{\sin\varphi(r^2 - 2rq_1\cos\varphi + q_1^2)},$$

$$r = \sqrt{q_1^2 - \alpha q_1 - \beta}, \quad \varphi = 2 \arctan \sqrt{\frac{2r + q_1 - \alpha}{2r - q_1 + \alpha}},$$

where q_1 can be computed by the formula of CARDANO for the reduced form of $P_3(x)$ (without the quadratic term).

3. CONVERGENT SEQUENCES $\{a_n\}$

In the two-dimensional case, that means $\gamma = 0$ in (1), we were able to characterize convergent sequences immediately from the real representation for a_n [2]. Some similar considerations yield in the three-dimensional case:

Theorem 1: The sequences $\{a_n\}$ defined by (1) are convergent if and only if the parameters α, β, γ are points of the three-dimensional region

(7) $\vartheta := \left\{(\alpha, \beta, \gamma) \in R^3 \mid \alpha + \beta + \gamma \leq 1,\ -\alpha + \beta - \gamma < 1,\ \gamma^2 - \alpha\gamma - \beta < 1\right\}$ (Fig. 1)

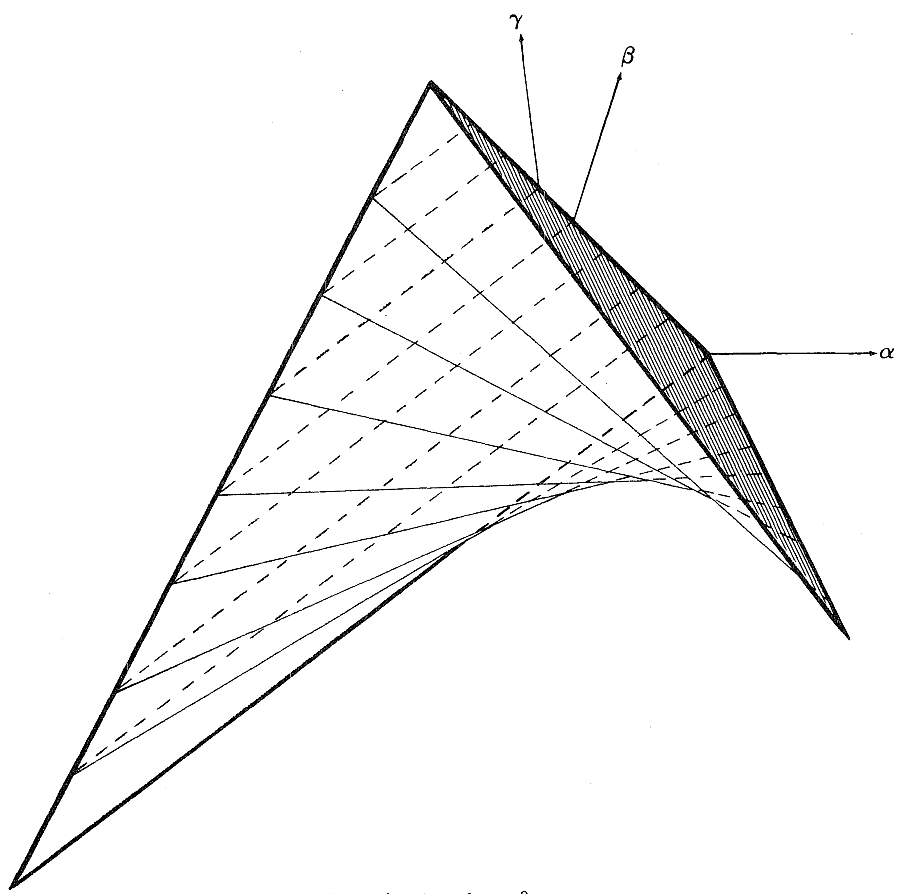

Fig. 1 Region ϑ

for all real numbers a_1, a_2, a_3. In the interior $\underline{\vartheta}$ of the region ϑ the sequences $\{a_n\}$ converge to zero. On the boundary $\alpha + \beta + \gamma = 1$ of ϑ the limit of a_n is given by

$$a := \lim_{n\to\infty} a = \frac{\gamma a_1 + (1 - \alpha)a_2 + a_3}{2 + \gamma - \alpha}, \quad 2 + \gamma - \alpha \neq 0. \tag{8}$$

Proof: From the real representations for a_n we obtain the following necessary and sufficient conditions for convergence.

1. All roots of $P_3(x) = 0$ are distinct and real:

$$-1 < q_1, q_2, q_3 \leq 1 \tag{9a}$$

2. Two distinct real roots:

$$-1 < q_1 \leq 1, \ -1 < q_2 = q_3 < 1 \tag{9b}$$

3. All roots are equal:

$$-1 < q_1 = q_2 = q_3 < 1 \tag{9c}$$

4. One real root and two conjugate complex ones:

$$-1 < q_1 \leq 1, \ 0 < q_2 q_3 = r^2 < 1. \tag{9d}$$

This means, for the polynomial $P_3(x)$, that

$$\begin{aligned} P_3(-1) &= -1 - \alpha + \beta - \gamma < 0, \\ P_3(1) &= 1 - \alpha - \beta - \gamma \geq 0. \end{aligned} \tag{10}$$

We have the following relations between the coefficients and the roots of $P_3(x)$ (VIETA):

$$\begin{aligned} q_1 + q_2 + q_3 &= \alpha, \\ q_1 q_2 + q_2 q_3 + q_1 q_3 &= -\beta, \\ q_1 q_2 q_3 &= \gamma. \end{aligned} \tag{11}$$

We start with the case $\gamma > 0$. Then q_1 may be the smallest of the positive roots, the only positive of the real roots, or the only real root of $P_3(x) = 0$. It follows from the last equation (11) with $0 < q_2 q_3 < 1$ from (9a)-(9d):

$$0 < \gamma < q_1.$$

We can conclude that, in the interval $[0, \gamma]$, there is no further root of $P_3(x)$; which, using the continuity of $P_3(x)$, means that $P_3(0)$ and $P_3(\gamma)$ have the same signs. So with $P_3(0) = -\gamma < 0$, $P_3(\gamma) = \gamma(\gamma^2 - \alpha\gamma - \beta - 1) < 0$, or with $\gamma > 0$,

$$\gamma^2 - \alpha\gamma - \beta < 1. \tag{12}$$

The case $\gamma = 0$ leads to the known two-dimensional case [2] and corresponds to the fact that one or more roots are zero. There we have convergence for points $(\alpha, \beta) \in R^2$ which satisfy the inequalities

$$\alpha + \beta \leq 1, \; -\alpha + \beta < 1, \; \beta > -1. \tag{13}$$

If $\gamma < 0$, then q_1 may be the greatest negative of the negative roots, the only negative of the real roots or the only real root of $P_3(x) = 0$. It follows from the last equation (11) with $0 < q_2 q_3 < 1$ that

$$q_1 < \gamma < 0.$$

We conclude, as in the first case, that $P_3(0)$ and $P_3(\gamma)$ have the same signs. We have with $P_3(0) = -\gamma > 0$, $P_3(\gamma) = \gamma(\gamma^2 - \alpha\gamma - \beta - 1) > 0$ or because of $\gamma < 0$

$$\gamma^2 - \alpha\gamma - \beta < 1. \tag{14}$$

So we have convergence in all cases if and only if $(\alpha, \beta, \gamma) \in R^3$ satisfy the inequalities (10), (12), (13), and (14), which define the required region ϑ (Fig. 1).

If (α, β, γ) are points of $\underline{\vartheta}$, the interior of ϑ, we have $|q_\nu| < 1$, $\nu = 1, 2, 3$, and it follows with the limits

$$\lim_{n \to \infty} n^\mu q_\nu^n = 0, \; \mu = 0, 1, 2; \; \nu = 1, 2, 3,$$

$$\lim_{n \to \infty} r^n = \lim_{n \to \infty} (q_2 q_3)^{\frac{n}{2}} = 0$$

from the real representation (3)-(6) for a_n, that a_n converges to zero. If $P_3(1) = 1 - \alpha - \beta - \gamma = 0$, we are on the boundary of ϑ (shaded area in Fig. 1). This means that 1 is a root of $P_3(x)$. We set $q_1 = 1$ and get, from (3), (4), or (6),

$$a: = \lim_{n \to \infty} a_n = A = \frac{\gamma a_1 - (\alpha - 1)a_2 + a_3}{2 + \gamma - \alpha} = \frac{(1 - \alpha - \beta)a_1 - (\alpha - 1)a_2 + a_3}{3 - 2\alpha - \beta} = G.$$

Also, if $q_2 = 1$, we have, from (11), $q_3 = \alpha - 2$, $2q_3 + 1 = -\beta$, and $q_3 = \gamma$, so that $\gamma^2 - \alpha\gamma - \beta = q_3 - (q_3 + 2)q_3 + 2q_3 + 1 = 1$, which contradicts the inequalities (12), (14); thus, $q_1 = 1$ must be a single root. Dividing $P_3(x)$ by the linear term $(x - 1)$, we get $P_2(x) := x^2 + (1 - \alpha)x + 1 - \alpha - \beta$. Since $q_1 = 1$ is a single root, we obtain $P_2(1) \neq 0$, so that $2 + \gamma - \alpha = 3 - 2\alpha - \beta \neq 0$, as stated in (8).

4. CONVERGENT SERIES

By the use of the generating function

$$\frac{a_1 + (a_2 - \alpha a_1)x + (a_3 - \alpha a_2 - \beta a_1)x^2}{1 - \alpha x - \beta x^2 - \gamma x^3} = \sum_{\nu=0}^{\infty} a_{\nu+1}x^{\nu} \tag{15}$$

we will give some limits of infinite series with a_ν, $\nu = 1, 2, \ldots$, as terms. First, we determine the radius of convergence ρ of the power series in (15). It is given by the smallest absolute value of the roots of

$$Q_3(x) := 1 - \alpha x - \beta x^2 - \gamma x^3 = 0. \tag{16}$$

Substituting in $Q_3(x) y = \frac{1}{x}$, $x \neq 0$, we get

$$Q_3\left(\frac{1}{y}\right) = \frac{1}{y^3}(y^3 - \alpha y^2 - \beta y - \gamma) = \frac{1}{y^3}P_3(y). \tag{17}$$

Using the notation of §3, with q_ν, $\nu = 1, 2, 3$, as the roots of $P_3(x)$ for the radius of convergence, we get

$$\rho = \min\left\{\frac{1}{|q_1|}, \frac{1}{|q_2|}, \frac{1}{|q_3|}\right\},$$

or as a further result,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{\rho} = \max\left\{|q_1|, |q_2|, |q_3|\right\}. \tag{18}$$

If $(\alpha, \beta, \gamma) \in \underline{\vartheta}$ we have $|q_\nu| < 1$, $\nu = 1, 2, 3$, so that

$$\rho > 1, \ (\alpha, \beta, \gamma) \in \underline{\vartheta}.$$

Especially, we have convergence in (15) for $x = 1$. So we get for $x = 1$:

$$\sum_{\nu=1}^{\infty} a_\nu = \frac{(1 - \alpha - \beta)a_1 + (1 - \alpha)a_2 + a_3}{1 - \alpha - \beta - \gamma}, \ (\alpha, \beta, \gamma) \in \underline{\vartheta}; \tag{19}$$

for $x = -1$:

$$\sum_{\nu=1}^{\infty} (-1)^{\nu-1} a_\nu = \frac{(1 + \alpha - \beta)a_1 - (1 + \alpha)a_2 + a_3}{1 + \alpha - \beta + \gamma}, \ (\alpha, \beta, \gamma) \in \underline{\vartheta}. \tag{20}$$

Addition or subtraction of (19) and (20) and division by 2 yields

$$(21)\quad \sum_{\nu=1}^{\infty} a_{2\nu-1} = \frac{[(1-\beta)^2 - \alpha^2 - 2\alpha\gamma]a_1 + (\gamma + \alpha\beta)a_2 + (1-\beta)a_3}{(1-\beta)^2 - (\alpha+\gamma)^2}, \quad (\alpha, \beta, \gamma) \in \underline{\vartheta}$$

or

$$\sum_{\nu=1}^{\infty} a_{2\nu} = \frac{\gamma(1-\beta)a_1 - [1 - \beta - \alpha(\alpha+\gamma)]a_2 + (\alpha+\gamma)a_3}{(1-\beta)^2 - (\alpha+\gamma)^2}, \quad (\alpha, \beta, \gamma) \in \underline{\vartheta}.$$

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