ON ODD PERFECT NUMBERS

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If $\sigma(n)$ denotes the sum of the positive divisors of a natural number n, and $\sigma(n) = 2n$, then n is said to be perfect. Elementary textbooks give a necessary and sufficient condition for an even number to be perfect, and to date 24 such numbers, 6, 28, 496, ..., have been found. (The 24th is

$$2^{19936}(2^{19937} - 1),$$

discovered by Bryant Tuckerman in 1971 and reported in the *Guiness Book of Records* [3]. The three preceding ones were given by Gillies [2].)

It is not known whether there are any odd perfect numbers, though many necessary conditions for their existence have been established. The most interesting of recent conditions are that such a number must have at least eight distinct prime factors (Hagis [4]) and must exceed 100^{200} (Buxton and Elmore [1]).

Suppose $p_1, \ldots, p_t$ are the distinct prime factors of an odd perfect number. In this note we will give a new and simple proof that

$$(1) \qquad \sum_{i=1}^{t} \frac{1}{p_i} < \log 2,$$

a result due to Suryanarayana [5], who also gave upper and lower bounds for

$$\sum_{i=1}^{t} \frac{1}{p_i}$$

when either or both of 3 and 5 are included in $\{p_1, \ldots, p_t\}$.

Most of these bounds were improved in a subsequent paper with Hagis [6], but no improvement was given for the upper bound in the case when both 3 and 5 are factors. We will prove here that in that case

$$\sum_{i=1}^{t} \frac{1}{p_i} < .673634,$$

the upper bound in [5] being .673770. We will also give a further improvement in the upper bound when 5 is a factor and 3 is not; namely,

$$\sum_{i=1}^{t} \frac{1}{p_i} < .677637,$$

the upper bound in [6] being .678036. (These are six-decimal-place approximations to the bounds obtained.)

We assume henceforth that n is an odd perfect number.

An old result, due to Euler, states that we may write

$$n = \prod_{i=1}^{t} p_i^{\alpha_i},$$

where $p_1, \ldots, p_t$ are distinct primes and $p_k \equiv \alpha_k \equiv 1 \pmod 4$ for just one k in $\{1, \ldots, t\}$ and $\alpha_i \equiv 0 \pmod 2$ when $i \neq k$. We will assume further that $p_1 < \cdots < p_t$, and later will commonly write $\alpha_{(r)}$ for α_i when $p_i = r$. The subscript k will always have the significance just given and Π' and Σ' will denote that $i = k$ is to be excluded from the product or sum.

We will need the well-known result

$$\frac{1}{2}(p_k + 1) \mid n, \tag{2}$$

which is easily proved (see [6]). It follows that

$$p_1 \le \frac{1}{2}(p_k + 1). \tag{3}$$

We also use the inequality

$$1 + x + x^2 > \exp\left(x + \frac{1}{4}x^2\right), \quad 0 < x \le \frac{1}{3}. \tag{4}$$

To prove this, note that

$$\exp\left(x + \frac{1}{4}x^2\right) - (1 + x + x^2) = 1 + x + \frac{x^2}{4} + \frac{1}{2!}\left(x + \frac{x^2}{4}\right)^2 + \cdots - (1 + x + x^2)$$

$$= -\frac{1}{4}x^2 + \frac{x^3}{4} + \frac{x^4}{32} + \frac{1}{3!}\left(x + \frac{x^2}{4}\right)^3 + \cdots,$$

so we wish to prove that

$$\frac{x}{4} + \frac{x^2}{32} + \frac{1}{3!x^2}\left(x + \frac{x^2}{4}\right)^3 + \frac{1}{4!x^2}\left(x + \frac{x^2}{4}\right)^4 + \cdots < \frac{1}{4}, \quad 0 < x \le \frac{1}{3}.$$

Now,

$$\frac{x}{4} + \frac{x^2}{32} \le \frac{1}{12} + \frac{1}{288} < .09$$

and

$$\frac{1}{3!x^2}\left(x + \frac{x^2}{4}\right)^3 + \frac{1}{4!x^2}\left(x + \frac{x^2}{4}\right)^4 + \cdots$$

$$< \frac{1}{6x^2}\left(x + \frac{x^2}{4}\right)^3 1 + \left(x + \frac{x^2}{4}\right) + \left(x + \frac{x^2}{4}\right)^2 + \cdots$$

$$\le \frac{1}{18}\left(\frac{13}{12}\right)^3 \frac{36}{23} < .12.$$

Hence (4) is true. Other and better inequalities of this type can be established but the above is sufficient for our present purposes.

Now we prove (1). Since n is perfect,

$$2n = \sigma(n) = \prod_{i=1}^{t}(1 + p_i + p_i^2 + \cdots + p_i^{\alpha_i})$$

so

$$2 = \prod_{i=1}^{t}\left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots + \frac{1}{p_i^{\alpha_i}}\right)$$

By Euler's result, $\alpha_k \ge 1$ and $\alpha_i \ge 2$ $(i \ne k)$, so

$$2 \ge \left(1 + \frac{1}{p_k}\right)\prod_{i=1}^{t}{}'\left(1 + \frac{1}{p_i} + \frac{1}{p_i^2}\right) > \left(1 + \frac{1}{p_k}\right)\prod_{i=1}^{t}{}'\exp\left(\frac{1}{p_i} + \frac{1}{4p_i^2}\right),$$

by (4). Hence,

$$\log 2 > \log\left(1 + \frac{1}{p_k}\right) + \sum_{i=1}^{t}{}'\left(\frac{1}{p_i} + \frac{1}{4p_i^2}\right)$$

$$> \frac{1}{p_k} - \frac{1}{2p_k^2} + \sum_{i=1}^{t}{}' \frac{1}{p_i} + \frac{1}{4}\sum_{i=1}^{t}{}' \frac{1}{p_i^2} > \sum_{i=1}^{t} \frac{1}{p_i} + \frac{1}{4p_1^2} - \frac{1}{2p_k^2}$$

$$\geq \sum_{i=1}^{t} \frac{1}{p_i} + \frac{1}{(p_k+1)^2} - \frac{1}{2p_k^2} > \sum_{i=1}^{t} \frac{1}{p_i}$$

using (3).

We end with the

Theorem: (i) If $15|n$, then

$$\sum_{i=1}^{t} \frac{1}{p_i} < \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \frac{1}{61} + \log \frac{2950753}{2815321} = a, \text{ say.}$$

(ii) If $5|n$ and $3\nmid n$, then

$$\sum_{i=1}^{t} \frac{1}{p_i} < \frac{1}{5} + \frac{1}{31} + \frac{1}{61} + \log \frac{293105}{190861} = b, \text{ say.}$$

Proof: The proofs consist of considering a number of cases which are mutually exclusive and exhaustive.

(i) We are given that $p_1 = 3$ and $p_2 = 5$. Suppose first that $\alpha_1 = 2$ and $\alpha_2 = 1$ (so that we are assuming, until the last paragraph of this proof, that $k = 2$). Since $\sigma(3^2) = 13$, we have $13|n$.

Suppose $\alpha_{(13)} = 2$, so that, since $\sigma(13^2) = 183 = 3 \cdot 61$, $61|n$. Since also $\sigma(5) = 6 = 2 \cdot 3$, we cannot have $\alpha_{(61)} = 2$, for $\sigma(61^2) = 3783 = 3 \cdot 13 \cdot 97$ and we would have $3^3|n$ (i.e., $\alpha_1 > 2$). Hence, $\alpha_{(61)} \geq 4$. Then, using a simple consequence of (4),

$$2 = \prod_{i=1}^{t}\left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots + \frac{1}{p_i^{\alpha_i}}\right)$$

$$> \left(1 + \frac{1}{3} + \frac{1}{3^2}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{13} + \frac{1}{13^2}\right)\left(1 + \frac{1}{61} + \frac{1}{61^2}\right.$$

$$\left. + \frac{1}{61^3} + \frac{1}{61^4}\right) \times \prod_{\substack{i=3 \\ p_i \neq 13,61}}^{t} \exp\left(\frac{1}{p_i}\right),$$

so, taking logarithms and rearranging,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log \frac{13}{9} - \log \frac{6}{5} - \log \frac{183}{169} - \log \frac{14076605}{13845841}$$

$$+ \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \frac{1}{61} = a.$$

If $\alpha_{(13)} \geq 4$, then we similarly obtain

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{3} + \frac{1}{3^2}\right) - \log\left(1 + \frac{1}{5}\right)$$

$$- \log\left(1 + \frac{1}{13} + \frac{1}{13^2} + \frac{1}{13^3} + \frac{1}{13^4}\right) + \frac{1}{3} + \frac{1}{5} + \frac{1}{13} < a.$$

Suppose now that $\alpha_1 \geq 4$ and $\alpha_2 = 1$. Then,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4}\right) - \log\left(1 + \frac{1}{5}\right) + \frac{1}{3} + \frac{1}{5} < a.$$

Next, suppose that $\alpha_2 \geq 5$. Then,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{3} + \frac{1}{3^2}\right)$$
$$- \log\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5}\right) + \frac{1}{3} + \frac{1}{5} < a.$$

Finally, suppose $k > 2$, so $\alpha_2 \geq 2$. Since $\alpha_k \geq 1$, we obtain, proceeding as above,

$$\log 2 > \log\left(1 + \frac{1}{p_k}\right) + \log\left(1 + \frac{1}{3} + \frac{1}{3^2}\right) + \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) + \sum_{i=3}^{t}{}' \frac{1}{p_i}$$
$$> \sum_{i=1}^{t} \frac{1}{p_i} + \log \frac{13}{9} + \log \frac{31}{25} - \frac{1}{3} - \frac{1}{5} - \frac{1}{2p_k^2}.$$

But $p_k \geq 13$ (though we can easily demonstrate that in fact $p_k \geq 17$), so,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log \frac{13}{9} - \log \frac{31}{25} + \frac{1}{3} + \frac{1}{5} + \frac{1}{338} < a.$$

This completes the proof of (i).

(ii) We are given that $p_1 = 5$. The details in the following are similar to those above. Suppose, until the last paragraph of this proof, that $\alpha_1 = 2$. Since $\sigma(5^2) = 31$, we have $31|n$. Now, $\sigma(31^2) = 993 = 3 \cdot 331$ and $3\nmid n$, so we must have $\alpha_{(31)} \geq 4$. It follows from (2) and from the fact that $3\nmid n$, that if $p_k < 73$, then p_k must be either 13, 37, or 61 (so we cannot have $\alpha_1 = 1$).

Suppose first that $p_k = 61$. Then $\alpha_{(61)} \geq 1$ and

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) - \log\left(1 + \frac{1}{31} + \frac{1}{31^2} + \frac{1}{31^3} + \frac{1}{31^4}\right)$$
$$- \log\left(1 + \frac{1}{61}\right) + \frac{1}{5} + \frac{1}{31} + \frac{1}{61} = b.$$

If $p_k = 13$, then, by (2), $p_2 = 7$. $\sigma(7^2) = 57 = 3 \cdot 19$, so $\alpha_2 \geq 4$, since $3\nmid n$. Also, $\alpha_{(13)} \geq 1$, so

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) - \log\left(1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4}\right)$$
$$- \log\left(1 + \frac{1}{13}\right) - \log\left(1 + \frac{1}{31} + \frac{1}{31^2} + \frac{1}{31^3} + \frac{1}{31^4}\right)$$
$$+ \frac{1}{5} + \frac{1}{7} + \frac{1}{13} + \frac{1}{31} < b.$$

If $p_k = 37$, then, by (2), $19|n$. $\sigma(19^2) = 381 = 3 \cdot 127$, so $\alpha_{(19)} \geq 4$. Since $\alpha_k \geq 1$,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) - \log\left(1 + \frac{1}{19} + \frac{1}{19^2} + \frac{1}{19^3} + \frac{1}{19^4}\right)$$
$$- \log\left(1 + \frac{1}{31} + \frac{1}{31^2} + \frac{1}{31^3} + \frac{1}{31^4}\right)$$
$$- \log\left(1 + \frac{1}{37}\right) + \frac{1}{5} + \frac{1}{19} + \frac{1}{31} + \frac{1}{37} < b.$$

If $p_k \geq 73$, then, as in the last paragraph of the proof of (i), we have

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2}\right) - \log\left(1 + \frac{1}{31} + \frac{1}{31^2} + \frac{1}{31^3} + \frac{1}{31^4}\right) + \frac{1}{5} + \frac{1}{31} + \frac{1}{2 \cdot 73^2} < b.$$

Finally, suppose $\alpha_1 \geq 4$. Then $p_k \geq 13$ and, as in the preceding paragraph,

$$\sum_{i=1}^{t} \frac{1}{p_i} < \log 2 - \log\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4}\right) + \frac{1}{5} + \frac{1}{2 \cdot 13^2} < b.$$

This completes the proof of (ii).

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A SIMPLE CONTINUED FRACTION REPRESENTS A MEDIANT NEST OF INTERVALS

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1. While working on some mathematical aspects of the botanical problem of phyllotaxis, I came upon a property of simple continued fractions that is simple, pretty, useful, and easy to prove, but seems to have been overlooked in the literature. I present it here in the hope that it will be of interest to people who have occasion to teach continued fractions. The property is stated below as a theorem after some necessary terms are defined.

2. Terminology: For any positive integer n, let $n/0$ represent ∞. Let us designate as a "fraction" any positive rational number, or 0, or ∞, in the form a/b, where a and b are nonnegative integers, and either a or b is not zero. We say the fraction is in lowest terms if $(a, b) = 1$. Thus, 0 in lowest terms is 0/1, and ∞ in lowest terms is 1/0.

If inequality of fractions is defined in the usual way, that is

$$a/b < c/d \text{ if } ad < bc,$$

it follows that $x < \infty$ for $x = 0$ or any positive rational number.