ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS *to* Professor A. P. Hillman, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. *Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.*

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-388 *Proposed by Herta T. Freitag, Roanoke, VA.*

Let T_n be the triangular number $n(n + 1)/2$. Show that

$$T_1 + T_2 + T_3 + \cdots + T_{2n-1} = 1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2$$

and express these equal sums as a binomial coefficient.

B-389 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Find the complete solution, with two arbitrary constants, of the difference equation

$$(n^2 + 3n + 3)U_{n+2} - 2(n^2 + n + 1)U_{n+1} + (n^2 - n + 1)U_n = 0.$$

B-390 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Find, as a rational function of x, the generating function

$$G_k(x) = \binom{k}{k} + \binom{k+1}{k}x + \binom{k+2}{k}x^2 + \cdots + \binom{k+n}{k}x^n + \cdots, \quad |x| < 1.$$

B-391 *Proposed by M. Wachtel, Zurich, Switzerland.*

Some of the solutions of $5x^2 + 1 = y^2$ in positive integers x and y are $(x, y) = (4, 9)$, $(72, 161)$, $(1292, 2889)$, $(23184, 51841)$, and $(416020, 930249)$. Find a recurrence formula for the x_n and y_n of a sequence of solutions (x_n, y_n) and find $\lim_{n\to\infty}(x_{n+1}/x_n)$ in terms of $a = (1 + \sqrt{5})/2$.

B-392 *Proposed by Phil Mana, Albuquerque, NM.*

Let $Y_n = (2 + 3n)F_n + (4 + 5n)L_n$. Find constants h and k such that

$$Y_{n+2} - Y_{n+1} - Y_n = hF_n + kL_n.$$

B-393 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

Let $T_n = \binom{n+1}{2}$, $P_0 = 1$, $P_n = T_1 T_2 \cdots T_n$ for $n > 0$, and $\begin{bmatrix} n \\ k \end{bmatrix} = P_n / P_k P_{n-k}$ for integers k and n with $0 \le k \le n$. Show that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{n - k + 1}\binom{n}{k}\binom{n+1}{k+1}.$$

SOLUTIONS

INCONTIGUOUS ZERO DIGITS

B-364 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX.*

Find and prove a formula for the number $R(n)$ of positive integers less than 2^n whose base 2 representations contain no consecutive 0's. (Here n is a positive integer.)

Solution by C. B. A. Peck, State College, PA.

Let S_n be the number of integers m with $2^{n-1} \leq m < 2^n$ and having a binary representation $B(m)$ with no consecutive pair of 0's. Clearly $S_n = R_n - R_{n-1}$ for $n > 1$ and $S_1 = R_1$. Also,

$$S_n = S_{n-1} + S_{n-2} \text{ for } n > 2,$$

since S_{n-1} counts the desired m for which $B(m)$ starts with 11 and S_{n-2} counts the desired m for which $B(m)$ starts with 101. It follows inductively that $S_n = F_{n+1}$, and then

$$R_n = S_1 + S_2 + \cdots + S_n = F_2 + F_3 + \cdots + F_{n+1} = F_{n+3} - 2.$$

Also solved by Michael Brozinsky, Paul S. Bruckman, Graham Lord, Bob Prielipp, A. G. Shannon, Sahib Singh, Rolf Sonntag, Gregory Wulczyn, and the proposer.

CONGRUENT TO A G.P.

B-365 *Proposed by Phil Mana, Albuquerque, NM*

Show that there is a unique integer $m > 1$ for which integers a and r exist with $L_n \equiv ar^n \pmod{m}$ for all integers $n \geq 0$. Also, show that no such m exists for the Fibonacci numbers.

Solution by Graham Lord, Université Laval, Québec.

Since $7 = L_4L_1 \equiv a^2r^5 \equiv L_2L_3 = 12 \pmod{m}$, then m divides 5, hence $m = 5$. Furthermore, $a = ar^0 \equiv L_0 = 2 \pmod{5}$. And finally, $ar^2 \equiv L_2 = L_1 + L_0 \equiv ar + a \pmod{5}$ together with $a \equiv 2 \pmod{5}$ implies $r^2 \equiv r + 1 \pmod{5}$, i.e., $r \equiv 3 \pmod{5}$. In all, $m = 5$, and a and r can be taken equal to 2 and 3, respectively. Note for any $n \geq 1$, $L_{n+1} = L_n + L_{n-1} \equiv ar^n + ar^{n-1} \equiv ar^{n+1} \pmod{5}$.

For the Fibonacci numbers, if m were to exist, then

$$3 = F_1F_4 \equiv a^2r^5 \equiv F_2F_3 = 2 \pmod{m},$$

i.e., $1 \equiv 0 \pmod{m}$, which is impossible if $m > 1$.

Also solved by George Berzsenyi, Paul S. Bruckman, Bob Prielipp, A. G. Shannon, Sahib Singh, Gregory Wulczyn, and the proposer.

LUCAS CONGRUENCE

B-366 *Proposed by Wray G. Brady, University of Tennessee, Knoxville, TN and Slippery Rock State College, Slippery Rock, PA.*

Prove that $L_iL_j \equiv L_hL_k \pmod{5}$ when $i + j = h + k$.

Solution by Paul S. Bruckman, Concord, CA and Sahib Singh, Clarion State College, Clarion, PA (independently).

Using the result of B-365,

$$L_i L_j - L_h L_k \equiv 2 \cdot 3^{i+j} - 2 \cdot 3^{h+k} \equiv 0 \pmod 5,$$

since $i + j = h + k$.

Also solved by George Berzsenyi, Herta T. Freitag, Graham Lord, T. Ponnudurai, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, and the proposer.

ROUNDING DOWN

B-367 *Proposed by Gerald E. Bergum, Sr., Dakota State University, Brookings, SD.*

Let $[x]$ be the greatest integer in x, $a = (1 + \sqrt{5})/2$ and $n \geq 1$. Prove that

(a) $F_{2n} = [aF_{2n-1}]$

and

(b) $F_{2n+1} = [a^2 F_{2n-1}]$.

Solution by George Berzsenyi, Lamar University, Beaumont, TX.

In view of Binet's formula,

$$aF_{2n-1} - F_{2n} = a\frac{a^{2n-1} - b^{2n-1}}{a - b} - \frac{a^{2n} - b^{2n}}{a - b} = -b^{2n-1}.$$

Similarly,

$$a^2 F_{2n-1} - F_{2n+1} = a^2 \frac{a^{2n-1} - b^{2n-1}}{a - b} - \frac{a^{2n+1} - b^{2n+1}}{a - b} = -b^{2n-1}.$$

Since $-1 < b = \dfrac{1 - \sqrt{5}}{2} < 0$ implies that $0 < -b^{2n-1} < 1$, the desired results follow.

Also solved by J. L. Brown, Jr., Paul S. Bruckman, Graham Lord, Bob Prielipp, A. G. Shannon, Sahib Singh, and the proposer.

CONVOLUTING FOR CONGRUENCES

B-368 *Proposed by Herta T. Freitag, Roanoke, VA.*

Obtain functions $g(n)$ and $h(n)$ such that

$$\sum_{i=1}^{n} iF_i L_{n-i} = g(n)F_n + h(n)L_n$$

and use the results to obtain congruences modulo 5 and 10.

Solution by Sahib Singh, Clarion State College, Clarion, PA.

Let $A_n = \sum_{i=1}^{n} iF_i L_{n-i}$. Then the generating function $A_1 + A_2 x + A_3 x^2 + \cdots$ is a rational function with $(1 - x - x^2)^3$ as the denominator. It follows that $g(n)$ and $h(n)$ are quadratic functions of n. Then, solving simultaneous equations for the coefficients of these quadratics leads to

$$g(n) = (5n^2 + 10n + 4)/10 \quad \text{and} \quad h(n) = n/10$$

so that

$$(5n^2 + 4)F_n + nL_n \equiv 0 \pmod{10}.$$

This also gives us $nL_n \equiv F_n \pmod 5$.

Also solved by Paul S. Bruckman, Graham Lord, Gregory Wulczyn, and the proposer.

NO LONGER UNSOLVED

B-369 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX.*

For all integers $n \geq 0$, prove that the set

$$S_n = \{L_{2n+1}, L_{2n+3}, L_{2n+5}\}$$

has the property that if $x, y \in S_n$ and $x \neq y$ then $xy + 5$ is a perfect square. For $n = 0$, verify that there is no integer z that is not in S_n and for which $\{z, L_{2n+1}, L_{2n+3}, L_{2n+5}\}$ has this property. (For $n > 0$, the problem is unsolved.)

Solution by Graham Lord, Université Laval, Québec.

That S_n has the property follows from the identities:

$$L_{2n+1}L_{2n+3} + 5 = L^2_{2n+2},$$

and

$$L_{2n+1}L_{2n+5} + 5 = L^2_{2n+3}.$$

In the second part of this solution use is made of the results:

① $2 \nmid L_{6k+1}$ and $2 \nmid L_{6k+5}$

② $4 = L_3 | L_{6k+3}$

③ $4 \nmid L_{2k}$

④ $4 \nmid F_{6k+3}$

Of these, ① is somewhat well known and the latter three are consequences of the results in "A Note on Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 2, No. 1 (February 1964), pp. 15-28, by L. Carlitz.

By ① and ② there is exactly one even number, L_{6k+3}, in the set S_n, $n \geq 0$. So if $\{z\} \cup S_n$ has the desired property, then $zL_{6k+3} + 5$ will be an odd square and thus congruent to 1 modulo 8. This implies that z, if it exists, is odd.

Now the other two members of S_n are either:

(a) L_{6k-1}, L_{6k+1}; (b) L_{6k+5}, L_{6k+7}; or (c) L_{6k+1}, L_{6k+5}.

Each of these is odd by ①, and hence the sum of 5 and any one of them multiplied by z will equal an even square. Thus, in case (a) [and similarly in case (b)]:

$$zL_{6k-1} + 5 \equiv 0 \pmod 4, \text{ and } zL_{6k+1} + 5 \equiv 0 \pmod 4;$$

i.e.,

$$zL_{6k} = z(L_{6k+1} - L_{6k-1}) \equiv 0 \pmod 4.$$

But this is impossible by ③ and the fact that z is odd.

And in case (c),

$$z \cdot 5F_{6k+3} = (zL_{6k+5} + 5) - (zL_{6k+1} + 5) \equiv 0 \pmod 4,$$

which is also impossible by ④.

Consequently, no z exists such that the set $\{z\} \cup S_n$ has the desired property. Note that it was not assumed that $n = 0$.

Also solved by Paul S. Bruckman, Herta T. Freitag, T. Ponnudurai, Bob Prielipp, A. G. Shannon, Sahib Singh, and the proposer.

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