FOLDED SEQUENCES AND BODE'S PROBLEM

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Readers of this journal have long been interested in Bode's Rule, see, e.g., [26] and [15]. Indeed attempts to solve it have ranged from those devoid of science but fairly accurate to those based on some physical principle(s) but rather inaccurate, such as Berlage's and O. Schmidt's theories. The problem of planetary motions was first tackled by Eudoxus, who proposed rotating tilted concentric spheres, rather like a gyroscope, to explain each planet. When Kepler solved the problem of their motions with the concept of areal velocity, the area swept out in an invariable plane divided by the time is a constant, the problem of a law for their spacing remained. Indeed, I think the unit of angular momentun should be named after Kepler for his contribution of area as a vector. Bode's problem is of great value to the history and especially the philosophy of science. The qualities that distinguish pure mathematics are succinctness, elegance, fertility, and relevance to the unsolved problems. But to a scientist the first criterion is reproducibility. The multifarious, variegated, and at times loquacious and mellifluous monographs on this aspect of cosmogony attest to man's persistent and insistent attempts, at times based on specious assumptions, to find order in a theatre of nature that may have no reason to be other than nearly random. Such is one view. But while science corrects its mistakes (so far) it must be remembered that Boltzmann committed suicide because his contemporaries would not accept his counting of molecules, Wagoner's 1911 theory of continental drift was not believed until the 'sixties, and Newton's theory was not believed on the continent until Clairaut's prediction of the return of Halley's comet in 1759 ($P = 76.75 \pm 1.5$ yr) came true. Such is the lag between prediction and proof. The final answer to Bode's problem will be known within twenty years when sophisticated computer simulations are finished. My work will probably remain the most accurate, namely 1-percent with a few exceptions either way. In any case, my work has led to some interesting mathematics, especially the Self-Lucas property (see Section 2). May it be that Urania and Euterpe have recessed a part of Nirvana to sequester all who have slaved over this vexing problem. The impetus for this paper comes from Kowal's [17] recent discovery of an object between Saturn and Uranus that *prima faciae*, see Table 1, fits my rule [6, 7] and does not fit any other rule published! Sequentially, I present an overview of the history of Bode's Rule, Kowal's discovery and then generalized folded sequences.

1. BODE'S PROBLEM AND KOWAL'S DISCOVERY

Historically, the first offered solution to Bode's problem was Kepler's [1] perfect solids, which model, in fact, antedates Bode by two centuries. Gingerich [11] has discussed the accuracy of Kepler's youthful proposal. At first, the Titius-Bode mnemonic was successful with the Asteroids and Uranus but it fails badly for Neptune and Mercury. Attempting to save it, Miss Blagg [3] introduced two more parameters into it. Nieto, see [9] of paper [7], supports her work. The literature is full of algebraic rules of the form, distance $\propto b^n$ and indeed of more complicated rules. It is interesting to look at the range of b values. A partial list is:

Dermott	$b = 2^{1/3} = 1.2599$ (Saturnian moons)
von Weizsacker	$b = 1.370889$ (10 eddies, inner planets),
Greig	$b = 1.378241$ (Saturn's inner moons)
Quadranacci	$b = 1.380$ (see below)
Dermott	$b = 3^{1/3} = 1.4422$ (Uranian moons)
Cale	$b = \phi/\sqrt{3} = 1.5115$ (see [15])
Pierucci & Dermott	$b = 4^{1/3} = 1.5874$ (Jovian moons)
Gaussin	$b = 1.72$
Blagg	$b = 1.73$
Dermott	$b = 6^{1/3} = 1.817121$
Belot	$b = 1.886$
von Weizsacker	$b = 1.894427$
Greig	$b = 1.8995476$
Quadranacci	$b = 1.905$ (see below)
Titius	$b = 2.00$

See Gould [15] for references I have not cited. Dermott (see [2] and [10] of paper [7]) was forced to take Earth and Venus together to retain his period factor of $\sqrt{6}$. Dermott's arbitrary period factor of $\sqrt{2}$ for Saturn's moons misses both Rhea and Janus.

Indeed, I have myself happened upon some rather well-fitting arbitrary rules. One such is a bisection of the Quadranacci recurrence $Q_{n+1} = Q_n + Q_{n-3}$, namely: 1, 1, 1, 1, 2, N = 3, 4, U = 5, 7, A = 10, 14, J = 19, ..., E = 95, ..., which is very good at representing reciprocal distances. Another one for the distances begins: 4, 7, 10, 15, ... The distance factor, b, is $(1.380278)^2 = 1.905166$. The most complex rule of which I am aware is Rothman's (see [9]), $d = n(5.5 + F_n^2)/9(1 + F_n)$. It has seven parameters: n, 5.5, 2, 9, and 1, and two to determine the Fibonacci sequence, and since only 9 planets are fitted then only 2 degrees of freedom are left which is unscientific. All of the above rules are arbitrary, except von Weizsacker's and my own. My own view is that any rule with more than *two* parameters violates Occkham's razor, "*Essentia non sunt multiplicanda praeter necessitatem.*"

Dermott (*ibid.*) proposed different period factors for each satellite system, whereas my theory is simpler since the same limiting ratio, ϕ^2, applies to all. He also ignored the outer Jovian and Saturnian moons. I chose to emphasize them. This suggests the principle of Contrary Ignorability: whatever earlier researchers ignore—that is the path to pursue. My work indicates that outer Jovian moons should cluster well within 10 percent of 97, 257, 730, and 608 days (Table 2 of [6]). The announcement of Jupiter's XIII moon [21] at 239 day and $i = 27°$ [25] came after my initial work [4], [22] and satisfies the above sequence. I also studied the relevance, if any, of rotation periods and grazing periods of parent bodies in [4], using:

$$(35) \qquad P_g^2 \rho_M = 3\pi/G \quad \text{where} \quad G = 498 \text{ day}^{-2}(\text{g/cc})^{-1}.$$

The period of a satellite just grazing the surface of Saturn, the Sun, Jupiter, and Uranus would be 0.167, 0.116, 0.12, and 0.11 day. The criterion for coalescence against tidal forces may be written

$$(36) \qquad \rho_m > fM/\text{distance}^3 \quad \text{or} \quad P_m > P_g\sqrt{(4\pi f \rho_M/3\rho_m)}$$

where m, M are the satellite and parent masses, ρ_m and ρ_M their densities, P_m the period of a satellite at this Roche limit, and f a factor between 2 and 10 [24, p. 18]. When (36) is not satisfied "rings" result. In [4] an inner Uranian moon was suggested which would have a period of 1/1.3292 = 0.752 day according to (17) of [6]. It has even been proposed that the separation, a, between binary stars satisfies Bode's rule [27], but I am very skeptical of that.

My own interest in Fibonacci numbers dates back at least to 1966 when I obtained a copy of Vorobyev's book. I tried these numbers on the planets with what seemed good accuracy and communicated this to Gould [13], pointing out the relevance of bisected Fibonacci sequences. After long arduous efforts, I thought I had put the problem to rest when news of Kowal's discovery [17] of a planetoid between Saturn and Uranus, too big to be a comet nucleus and too small to be a large planet, was announced. He calls the object Chiron after one of the Greek half-man/half-horse animals. Is this discovery to prognosticate that this Chinese year 4676 (see [18]), beginning 7 Feb. 1978, should not be the year of Earth-Horse, but rather the year of the Centaur!? One can see in Table 1 that Chiron fits very neatly into the bisected half-integer sequence. I could have predicted this object three years ago [5, 6] from the folded sequences I had discovered but it would have been considered wildly delusionary at the time. The major body that should occur before Neptune in my sequence given in [5] and Table 2 of [6] is easily calculated to have a reciprocal period corresponding to -1974 - 4558 = -6532. Its period should then be 317816/6532 = 48.66 yrs. The agreement with Chiron's period of 47 to 51 yrs is quite good. The ellipsis (...) in [5] and [6] clearly indicated that the sequence continued in both directions so that a body at Chiron's position was implied. In an earlier work [4] I had stated, ". . . one should really ask why don't Jupiter, Uranus and Neptune have a plane of particulate matter [rings] inside Roche's limit since that is natural considering the pervasiveness of grains and cometesimals . . ." (p. 16). As we now know, rings have been found around Uranus [16], [28]. The way to test my theory would be a computer simulation using reciprocal periods given by (23a) or (23b) or [7] (or, perhaps, using part of a bisected odd-N folded sequence in [6]) to see if the broad maxima in [10] can be sharpened.

The agreement of Chiron's period with Folded Fibonacci sequences is reassuring but not perfect. I have been able to represent Neptune and outer satellites in general more accurately than *any* other rule simply because I worked with recursive sequences rather than naive power laws. Also, mine is the only work to represent the several comet groups (see Table 1). So that, in terms of completeness and goodness of fit, my hypothesis is the best. It remains for a computer simulation to test whether my proposal gives maximum stability. Such a simulation may solve the following question: Why are some period ratios nearly but not quite small integers? Saturn:Jupiter is not 5:2 but is 6551:2638 to seven significant digits. Accurate to only five digits is 149:30. Neptune:Uranus is not 2:1 but is 51:26 to five digits. And Uranus:Saturn is 77:27 to nearly five digits. Similarly, Earth:Venus is not F_7:F_6 but 1172:721 to eight-digit accuracy. These ratios suggest that low-order commensurabilities (LOC) are avoided, except for the ratio 2 among the Galilean satellites. The Kirkwood gaps indicate that LOC are unstable if the ratio ≥ 2, such as 11/5, 9/4, 7/3, 5/2, 8/3, and 3/1.

The problem is ancient. The Pythagoreans believed in orbits in arithmetic progression and added a Central Fire and a Counter-Earth [23] to obscure that Fire so that the total number of moving bodies be the "magic" number 1 + 2 + 3 + 4 = 10. Yet Aristarchus placed the Sun in the center for reasons of simplicity 18 centuries before Copernicus. Later, Ptolemy and others confounded the picture with equants and epicycles, until Kepler discovered that blemished curve—the ellipse.

Further back in time, the concepts become anthropocentric and folklorish as in the mural of Ra and Noot found in the tomb of Rameses VI.

There is still the possibility that Bode's problem has no solution or that the distribution of planets is random on a logarithmic axis, save that they cannot be too close to each other. But my work has led to some interesting sequences that I will discuss in the future.

TABLE 1

The Correspondence between the Half-Integer Sequence and the Planets

Reciprocal Period	Period (yrs)	Solar System
-55Ø - 233	0.999969	
34Ø + 144	1.618	null
-21Ø - 89	2.6183	Hungaria #434 (991 da = 2.71 yrs), etc. (Average = 2.75 yrs)
13Ø + 55	4.235	null
-8Ø - 34	6.859	Faye (7.35 yrs); Brooks II (6.72 yrs); d'Arrest (6.67 yrs), Finlay (6.90 yrs), etc.
5Ø + 21	11.07	null
-3Ø - 13	18.035	Neujmin (17.97 yrs)
2Ø + 8	28.66	null
-Ø - 5	48.66	Chiron (47 to 51 yrs), other Centaurs
Ø + 3	69.73	Olbers (69.6 yrs); Brorson-Metcalf (69.1 yrs); Pons-Brooks (71 yrs); Halley
- 2	161.00	Neptune (164.79 yrs); $N + P$ (168.4 yrs)
Ø + 1	123.0	Swift-Tuttle (119.6 yrs); Barnard II (128.3 yrs)
Ø - 1	521.0	Planet X (464.? yrs)
2Ø	99.5	null
3Ø - 1	83.55	Uranus (84.01 yrs)
5Ø - 1	45.42	null
8Ø - 2	29.42	Saturn (29.46 yrs)
13Ø - 3	17.85	null
21Ø - 5	11.11	(Jupiter 11.86 yrs) not meant to fit Jupiter.
34Ø - 8	6.849	null
55Ø - 13	4.237	Astrea (4.13 yrs); asteroids (0.23 yr^{-1})
89Ø - 21	2.6177	null
144Ø - 34	1.618	(Mars)
233Ø - 55	1.0	(Earth)
377Ø - 89	0.618	(Venus, 0.615 yr)
610Ø - 144	0.382	null
987Ø - 233	0.236	(Mercury, 0.241 yr)

2. GENERALIZED FOLDED SEQUENCES

The obvious generalization of the definition of folded sequences, (4) of [6], is

$$\{\mathcal{P}_{j,N}\}_k = P_{j,k} + (-1)^{N+1} P_{j,k-N} \quad \text{with} \quad 0 \le k \le N - 1 \tag{37}$$

where a *script* letter denotes a folded sequence, $\{P_j\}$ is the jth coprime sequence as in (1) of [20], and $\{\mathcal{P}_{j,N}\}$ is finite if N is finite. This latter point differs from (1) of [6] wherein folded sequences were made infinite by repeating the cycle *ad infinitum*. It will help to display the Folded Pell array $\{\mathcal{P}_2\}$ in Table 2:

TABLE 2

Folded Pell Sequences for N Odd

N													Sum
1						1							1
3					5	-1	3						7
5				29	-11	7	3	13					41
7			169	-69	31	-7	17	27	71				239
9		985	-407	171	-65	41	17	75	167	409			1393
11	5741	-2377	987	-403	181	-41	99	157	413	983	2379		8119
13	-13859	5743	-2373	997	-379	239	99	437	973	2383	5739	13861	47321
15	...	-13855	5753	-2349	1055	-239	577	915	2407	5729	13865	...	
17			...	5811	-2209	1393	577	2547	5671	13889	...		
∞			...	$6+5\sqrt{8}$	$-2-2\sqrt{8}$	$2+\sqrt{8}$	$2+0$	$6+\sqrt{8}$	...				
∞				...	$6+\sqrt{8}$	$-2-0$	$2+\sqrt{8}$	$2+2\sqrt{8}$	$6+5\sqrt{8}$	...			
r		...	$-7h$	$-5h$	$-3h$	$-h$	h	$3h$	$5h$	$7h$	...		

The last row is the half-integer subscript r defined by $2k = 2r + N$. Both infinite folded sequences (which I often call half-integer sequences) are given, namely for $N = 1 \pmod 4$ and $N = 3 \pmod 4$. Subscripts within braces are part of the *name* of the array/sequence, whereas those outside the braces indicate the value of the row/element.

In order to find the jth Half-Integer sequence the theorem in [6] is generalized.

Theorem: $\{\mathcal{P}_{j,N}\}_{k+1} / \{\mathcal{P}_{j,N}\}_k$ approaches a limit as $N \to \infty$ for each k dependent only upon the value of N modulo 4.

Proof: Define $\theta = N$ modulo 4. Since $r = (k - N/2)$ then $r = -h$ gives

$$k = (N - 1)/2 = [N/2].$$

We will need $P_{-k}P_{-k-1}/(P_{k+1}P_k) = -1$, from which one finds

$$P_{-k}/P_{k+1} = -z = -P_k/P_{-k-1}.$$

Thus $z \to \pm\beta$ as $N \to \infty$, while $\theta = 3$ or 1, respectively. Then,

$$\{\mathcal{P}_{j,N}\}_h / \{\mathcal{P}_{j,N}\}_{-h} = (P_{k+1} + P_{-k})/(P_k + P_{-k-1})$$
$$= i^{\theta-1}(1 - z)/(1 + z)$$

which $\to \pm((\alpha - 1)/(\alpha + 1))^{\pm 1}$ as $N \to \infty$, while $N = 1$ or $3 \pmod 4$, respectively. Hence, where the limit of $\mathcal{P}$ is $\mathcal{S}$,

(38) $$\{\mathcal{S}_j\}_{-h}/\{\mathcal{S}_j\}_h = (\alpha + 1)/(\alpha - 1) = -\{\mathcal{S}_j^*\}_h/\{\mathcal{S}_j^*\}_{-h}.$$

In the following the first subscript of $\mathcal{S}_{j,r}$ and $P_{j,n}$ is suppressed.

A number of relations follow from the elegant

(39) $$\mathcal{S}_r = P^*_{r+h} + dP_{r-h} \quad \text{and} \quad \mathcal{S}^* = P^*_{r-h} + dP_{r+h};$$

these are (40), (41), (42), (43), and (45).

(40) $jP_m^* = (\mathcal{S}_{m+h} - \mathcal{S}_{m-h}^*)$ and $P_n = (\mathcal{S}_{n+h}^* - \mathcal{S}_{n-h})/jd$,

(41) $(2 + d)P_{r+h}^* = \mathcal{S}_{r+1}^* + \mathcal{S}_r$ and $d(d + 2)P_{r-h} = (\mathcal{S}_r + \mathcal{S}_{r-1}^*)$,

(42) $(\mathcal{S}_r + \mathcal{S}_r^*) = 2\zeta\alpha^r$ and $(\mathcal{S}_r - \mathcal{S}_r^*) = 2\zeta i\beta^r$,

and the Binet-like formulas

(43) $\mathcal{S}_r = \zeta(\alpha^r + i\beta^r)$ and $\mathcal{S}_r^* = \zeta(\alpha^r - i\beta^r)$

where

(44) $\zeta = (\alpha^h + i\beta^{-h}) = (\alpha^h + \alpha^{-h}) = -i(\beta^h - \beta^{-h})$, where $i = \sqrt{-1}$,

and analogous to $F_n = (-1)^{n+1}F_{-n}$, we have

(45) $\mathcal{S}_r^* + \mathcal{S}_r = (\mathcal{S}_{-r} - \mathcal{S}_{-r}^*)i^{r+h}$.

Now the initial values are determined by (37) or (38) and are

(46) $\mathcal{S}_{j,-h} = (2 + d) = \mathcal{S}_{j,h}^*$ and $\mathcal{S}_{j,h} = j = -\mathcal{S}_{j,-h}^*$ for all j.

The bisections of the general Half-Integer sequence for $N = 1 \pmod 4$ appear in (47) and (48), where t instead of j is used from now on for the parameter. Compare with the penultimate rows of Table 2. Also (49) is the subscript r. In Table 3 note that $(2 + d) = \zeta^2 = (\alpha^h + \alpha^{-h})^2$ and $v = \sqrt{5}$.

(47)	$(t^4+dt^4+4t^2+3dt^2+d+2)$		(t^2+dt^2+d+2)		$2+d$			(t^2+2+d)
(48)		$-(t^3+dt^3+3t+2dt)$		$-(t+dt)$		t		$(t^3+3t+dt)$
(49)	$-\frac{9}{2}$	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$

TABLE 3

Parameters of Half-Integer t-Fib Sequences

t	d	α	I_{-h}/I_h	$(\alpha^h+\alpha^{-h})$	$(\alpha^h-\alpha^{-h})$	td
$\sqrt{v-2}$	$\sqrt{v+2}$	1.272020	8.352410	2.014490	0.241187	1.000
$1/\sqrt{2}$	$3/\sqrt{2}$	1.414213	5.828427	2.030104	0.348311	1.500
1	$\sqrt{5}$	1.618034	4.236068	2.058171	0.485868	2.236
1.5	5/2	2.0	3.0	2.121320	0.707107	3.750
2	$\sqrt{8}$	2.414213	2.414213	2.197368	0.910180	5.657
$\sqrt{5}$	3	2.618034	2.236068	2.236068	1.0	6.708
8/3	10/3	3.0	2.0	2.309401	1.154700	8.888
3	$\sqrt{13}$	3.302776	1.868517	2.367605	1.267103	10.817
$2\sqrt{3}$	4	3.732050	1.732050	2.449490	1.414213	13.856
4	$\sqrt{20}$	4.236068	1.618034	2.544039	1.572303	17.888
$\sqrt{32}$	6	5.828427	1.414214	2.828427	2.000000	33.941

Table 3 shows us that the Half-Integer Pell sequence is the only one in which the ratio of the central pair, S_{-h}/S_h equals the characteristic root. This is clear from $(\alpha + 1)/(\alpha - 1) = \alpha$ whose solution is $1 + \sqrt{2}$. Sequences given by $(\alpha^h + \alpha^{-h}) = \alpha$ have a very simple Binet-like formula but the larger root $\alpha = 2.1478990$ is the solution to a cubic. Are there Half-Integer sequences with integer terms? Consider t-Fib sequences,

$$P_{t,n+1} = tP_{t,n} + P_{t,n-1},$$

for which d is an integer, $d^2 = t^2 + 4$. Consider the Root-Five sequences, $t = \sqrt{5}$, which have the FL-types: 1, 0, 1, v, 6, $7v$, 41, $48v$, 281, $(7 \cdot 47)v$, $(18 \cdot 107)$, $(55 \cdot 41)$, ... and 2, v, 7, $8v$, 47, $55v$, $(23 \cdot 14)$, $377v$, 2207, ... where $v = \sqrt{5}$. In the corresponding Half-Integer sequence, from $r = -11h$ to $+11h$,

(50) $\quad \ldots, -199v, 170, -29v, 25, -4v, 5, v, 10, 11v, 65, 76v, 445, \ldots$

we see that both bisections are integers (after dividing by common factors). Furthermore, one bisection consists of every fourth Fibonacci number including F_5 and the other consists of every fourth Lucas number including L_5. Can this be generalized? As a little algebra shows, yes,

(51) $\quad d_i = d_4^2 - 2 \quad \text{or} \quad d_i = t_4^2 + 2 \quad \text{or} \quad t_i = t_4 d_4$

where t_4, d_4 refer to the sequence from which every fourth term is extracted and d_i, t_i refer to the chosen sequence. To illustrate this, the Root-32 Half-Integer sequence, see Table 3, has from (46)

$$S_h = t = 4\sqrt{2} \quad \text{and} \quad S_{-h} = 2 + d = 8.$$

The bisections of this reduced by common factors are:

(52) $\quad \ldots, 33461, 985, 29, 1, 5, 169, 5741, \ldots$

(53) $\quad \ldots, -8119, -239, -7, 1, 41, 1393, 47321, \ldots$

which are every fourth Pell number as expected beginning with $P_5 = 29$ and $P_5^*/2 = 41$. Proofs of statements above follow easily from (39) or (43). So (23a, b) are every fourth term of the $t = \sqrt{(\sqrt{5} - 2)}$ sequences. The "F" sequence to 3 decimals is ..., 1.236, -0.486, 1, 0, 1, 0.485868, 1.236, 1.086, 1.764, 1.943, ... and clearly the ratios

$$1.236:1:1.764 = 3 + v:2 + v:3 + 2v = 2\emptyset:\emptyset + 1:\emptyset + 3$$

show that every fourth term of "F" gives (23a) of [7] and equivalently a bisection of Table 1. The general recurrence of these bisections is

$$B_{t,n+1} = P^*_{t,4}B_{j,n} - B_{t,n-1}$$

where $P^*_{2,4} = 34$ for (52) and (53). A bisected t-Fib sequence has the recurrence

$$P_{n+2} = (t^2 + 2)P_n - P_{n-2} \quad \text{or} \quad \delta^2 P_n = t^2 P_n.$$

Indeed the recurrence's middle term for m-sectioning has a coefficient given by the mth rising diagonal of the Lucas triangle. The bi-bisection case is $P_{n+4} = (a^4 + 4a^2b + 2b^2)P_n - b^4P_{n-4}$, and so on.

We come now to what I regard as the most important property of these sequences. The Self-Lucas property, (14) of [6], remains unchanged in this generalization, namely

(54) $\quad (\mathcal{S}_{r+1} + \mathcal{S}_{r-1})/d = (-1)^{r-h}\mathcal{S}_{-r} \quad \text{and} \quad (\mathcal{S}^*_{r+1} + \mathcal{S}^*_{r-1})/d = (-1)^{r+h}\mathcal{S}^*_{-r},$

where $d = (\alpha - \beta)$ and $(\alpha + \beta) = j = t$. Now (54) may be proven from (43). Note that terms with subscripts $(r + 1)$, $(r - 1)$ and $-r$ all belong to the *same* bisection of $\mathcal{S}_t$ or $\mathcal{S}_t^*$. This is obvious since $(r + 1) - (r - 1) = 2$ and $(r - 1) - (-r) = 2k - (N + 1)$ which is also even. Taking ratios of (54) one may form the triplet rule, for both $\mathcal{S}_t$ and $\mathcal{S}_t^*$,

(55) $$(\mathcal{S}_r + \mathcal{S}_{r+2})/(\mathcal{S}_r + \mathcal{S}_{r-2}) = \mathcal{S}_{-r-1}/\mathcal{S}_{-r+1}.$$

Now this is readily illustrated when d is an integer so consider (50). Obviously $(10 + 65)/3 = 25$ and $(11v + 76v)/3 = 29v$ and both are members of (50). Again from (52) one has $(985 + 29)/6 = 169$. From Table 1, I illustrate (55) by $((8\emptyset - 2) + (3\emptyset - 1))/((3\emptyset - 1) + (\emptyset - 1)) = (\emptyset + 5)/2$ by using the trick $(\emptyset + 2) = \emptyset v$. But this last ratio, $(\emptyset + 5):2$ is Chiron:Neptune.

I introduce a new operator capital lambda, Λ:

(56) $$\Lambda \equiv I + E,$$

where E and I are the forward shift and identity operators and, therefore, $(\Lambda - \nabla) \equiv (E + E^{-1})$. Then the Self-Lucas property may be written

(57) $$(\Lambda - \nabla)\mathcal{S}^{\pm}_{t,r} = d(-1)^{r \mp h}\mathcal{S}^{\pm}_{t,-r},$$

where $\mathcal{S}^{\pm}$ is another notation for $\mathcal{S}$ and $\mathcal{S}^*$, respectively. We may also write $\Lambda B_n = dB_{-n}$ or $-dB_{-n}$ depending upon which bisection of $\mathcal{S}_t$ or $\mathcal{S}_t^*$ is being considered. The Self-Lucas property does not hold for F-like sequences: $P_{t,1} = 1 = P_{t,-1}$, or L-like sequences: $P^*_{t,1} = 1 = -P^*_{t,-1}$. In this sense, the Half-Integer sequences are more important.

How are (54) and (57) to be interpreted geometrically? Let alternate terms of the bisection be made negative, then the recurrence is

(58) $$\delta^2 N_n = -(t^2 + 4)N_n.$$

Further let the terms of N_n be reciprocal periods of planets and let a minus sign mean retrograde (backward) motion. Then the Self-Lucas property may be written

(59) $$\Delta N_n = -dN_{-n},$$

which in words says that the set of synodic (apparent) frequencies of a collection of alternately pro- and retrograde planets are simply proportional to the negative of the sidereal (real) frequencies in reverse order.

3. COMMENTS ON THE RECIPROCAL PERIOD RULE

Why are the planetary frequencies not Folded or Half-Integer *Pell* sequences? The limiting distance ratio would be 3.2386766. One solution to this is point (x) of [7], namely to bring the planets closer to each other, thereby minimizing their potential energy which is negative; that is,

(60) $$\max \sum_{i \neq j} \left(GM_J m_i / d_{av}^2\right),$$

where M_J is Jupiter's mass, m_i the mass of any planet except Jupiter, and d_{av} is a time-weighted distance from Jupiter. The Pell and, indeed, all t-Fib sequences satisfy point (ix), the avoidance of low-order commensurabilities, since $\gcd(P_{t,n+1}, P_{t,n}) = 1$ for all integers t and n. This can also be seen by noting that the continued fraction of the roots of any t-Fib recurrence consists of repeated $(1/t)$'s, so no one convergent is a great deal better than another. The sequence, 11, 12, 16, 24, 38, ..., is an example where gcd(12, 16) $\neq$ 1. Given that the total number of planets is a constant, then minimization of the cumulative perturbation frequencies (synodic) occurs as t becomes small (point xii). Of course, as t becomes small the average distance becomes smaller and the average perturbation force becomes very large.

4. FINALE

The logic [7] of my rule suggests that other civilizations may be signaling us in binary code with $1/\emptyset^2$ = 0.01100,00111,00100,01000,01100, ... But let me assure you that my getting into Bode's Rule was not a matter of choice. Its rewards, though, have been a large number of empyreal highs, some over ideas I later rejected; but now I am glad to be through with this whirlpool. Finally, we all know that the idea of the "music of the spheres" which dates back to Eudoxus is poetic license, nonetheless I could not help noting that though most of the "notes" in my scale are cacophpnous, the first note, $2+v$, corresponds to C-sharp two octaves high, since $(2+v) \cong 2^{25/12}$.

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FIBONACCI NUMBERS IN COIN TOSSING SEQUENCES

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The Fibonacci numbers and their generating function appear in a natural way in the problem of computing the expected number [2] of tosses of a fair coin until two consecutive heads appear. The problem of finding the expected number of tosses of a p-coin until k consecutive heads appear leads to classical generalizations of the Fibonacci numbers.

First consider tossing a fair coin and waiting for two consecutive heads. Let O_n be the set of all sequences of H and T of length n which terminate in HH and have no other occurrence of two consecutive heads. Let S_n be the number of sequences in O_n. Any sequence in O_n either begins with T, followed by a sequence in O_{n-1}, or begins with HT followed by a sequence in O_{n-2}. Thus,

$$S_n = S_{n-1} + S_{n-2},\ S_1 = 0,\ S_2 = 1. \tag{1}$$

Consequently, $S_{n-2} = F_n$, the nth Fibonacci number. The probability of termination in n trials is $S_n/2^n$. Letting

$$g(x) = \sum_2^\infty S_n x^n,$$

and using the generating function $(1 - x - x^2)^{-1}$ for the Fibonacci numbers, yields $g(x) = x^2/(1 - x - x^2)$. Hence, the expected number of trials is

$$\sum_{n=1}^\infty nS_n/2^n = (1/2)g'(1/2) = 6.$$

We generalize this result to the following

Theorem: Consider tossing a p-coin, $Pr(H) = p$, repeatedly until k consecutive heads appear. If P_n is the probability of terminating in exactly n trials (tosses), then the generating function

$$G(x) = \sum_k^\infty P_n x^n \text{ is given by } G(x) = \frac{(px)^k(1 - px)}{1 - x + \dfrac{(1-p)}{p}(px)^{k+1}} \tag{2}$$

The expected number of trials, $G'(1)$ is

$$1/p + 1/p^2 + \cdots + 1/p^k = \frac{1}{1-p}\left[\frac{1}{p^k} - 1\right]. \tag{3}$$

Proof: Let O_n be the set of all sequences of H and T of length n which terminate in k heads and have no other occurrence of k consecutive heads. Let S_n be the number of sequences in O_n and $P_n = Pr(O_n)$ be the probability of the event O_n. One possibility is that a sequence in O_n begins wi h a T, followed by a sequence in O_{n-1}; the probability of this is