ON THE DENSITY OF THE IMAGE SETS OF CERTAIN ARITHMETIC FUNCTIONS—III

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1. INTRODUCTION

Let n be a fixed but arbitrary nonnegative integer. It is known (see [1], for example) that n may be uniquely represented in the form $n = d_1 1! + d_2 2! + \cdots + d_k k!$, $0 \le d_j \le j$. Suppose that $f(d, j)$ is a nonnegative integer-valued function of j for each "digit" d, $0 \le d \le j$, $j = 1, 2, \ldots$, and define

$$S(n) = \sum_{j=1}^{k} f(d_j, j),$$

$$T(n) = n + S(n),$$

$$\Omega(k, r) = \{T(x) \mid k \le x \le r\},$$

$$D(k, r) = |\Omega(k, r)|$$

$$\Omega(r) = \Omega(0, r)$$

$$D(r) = D(0, r)$$

$$\mathcal{R} = \{x \mid x = T(n) \text{ for some } n\}, \text{ and}$$

$$\mathcal{C} = \{x \mid x \ne T(n) \text{ for any } n\}.$$

Our objective here is to prove some results concerning the asymptotic density of the sets $\mathcal{R}$ and $\mathcal{C}$ analogous to those which we proved when we considered the representation of n as an integer in base b (see [2] and [3]).

2. EXISTENCE AND COMPUTABILITY OF THE DENSITY

Theorem 2.1: Let $f(d, j)$, $0 \le d \le j$ be as described above. If

(a) $f(0, j) = 0$, $j = 1, 2, \ldots$
(b) $f(d, j) = o(j!)$ uniformly in j, i.e.,
$\sup \{f(d, j), 0 \le d \le j\} = o(j!)$

then the density of $\mathcal{R}$ exists.

Proof: We first show that

(2.2) $$D(dk!, dk! + r) = D(r), \quad 0 \le r \le k! - 1.$$

To prove 2.2, let us suppose that

$$x = dk! + \sum_{j=1}^{k-1} d_j j! \quad \text{and} \quad y = dk! + \sum_{j=1}^{k-1} d_j' j!.$$

Clearly, $T(x) = T(y)$ if and only if

$$T\left(\sum_{j=1}^{k-1} d_j j!\right) = T\left(\sum_{j=1}^{k-1} d_j' j!\right).$$

Suppose that $d_{k-1} = d_{k-2} = \cdots = d_{k-t} = 0$ (or that $d'_{k-1} = d'_{k-2} = \cdots = d'_{k-t} = 0$). Since $f(0, j) = 0$, it must be the case that

$$T\left(\sum_{j=0}^{k+t-1} d_j j!\right) = T\left(\sum_{j=0}^{k-1} d_j j!\right) = T\left(\sum_{j=0}^{k-1} d'_j j!\right).$$

We have therefore exhibited a one-one correspondence between the elements of $\Omega(dk!, dk! + r)$ and $\Omega(r)$, $0 \le r \le k! - 1$, and hence 2.2 follows. In particular, if $r = k! - 1$, we obtain

(2.3) $$D(dk!, (d + 1)k! - 1) = D(k! - 1).$$

Our next result will enable us to find a relationship between

$$D((k + 1)! - 1) \quad \text{and} \quad \sum_{d=0}^{k+1} D(dk! - 1).$$

Lemma 2.4: There exists an integer k_0 such that for all $k \ge k_0$ the sets $\Omega(0, k! - 1)$, $\Omega(k!, 2k! - 1)$, ..., $\Omega(kk!, (k + 1)! - 1)$ are pairwise disjoint, except possibly for adjacent pairs.

Proof: The maximum value in $\Omega(dk!, (d + 1)k! - 1)$ is at most $(d + 1)k! - 1 + kM_k$, where $M_k = \max\{f(d, j), 1 \le j \le k\}$, and the minimum value in $\Omega((d + 2)k!, (d + 3)k! - 1)$ is at least $(d + 2)k!$. By assumption (b), there exists k'_0 such that $f(d, j) < j!/2$, for all $j \ge k'_0$, and there exists $k_0 \ge k'_0$ such that $f(d, j) < j!/2 - k'_0 M_{k'_0}$, for all $j \ge k_0$, where $M_{k'_0} = \max\{f(d, j) \mid 1 \le j \le k'_0\}$. Therefore, if $k \ge k_0$, we have

$$\sum_{j=1}^{k} f(d_j, j) = \sum_{j=1}^{k'_0} f(d_j, j) + \sum_{j=k'_0+1}^{k_0} f(d_j, j) + \sum_{j=k_0+1}^{k} f(d_j, j) < k'_0 M_{k'_0} + \sum_{j=k'_0+1}^{k} j!/2 - k'_0 M_{k'_0}(k - k_0) \le \sum_{j=k'_0+1}^{k} j!/2 < k!.$$

In particular, $kM_k < k!$ if $k \ge k_0$. Hence, we certainly have $(d + 1)k! - 1 + kM_k < (d + 2)k!$ if $k \ge k_0$, so the result is proved.

Now let $\lambda_{d,k} = |\Omega(dk!, (d + 1)k! - 1) \cap \Omega((d + 1)k!, (d + 2)k! - 1)|$, $0 \le d \le k - 1$. Using 2.3 and 2.4 and the fact that

$$D((k + 1)! - 1) = \sum_{d=0}^{k} D(dk!, (d + 1)k! - 1) - Q,$$

where Q depends on the number of elements that the sets $\Omega(0, k! - 1)$, $\Omega(k!, 2k! - 1)$, ..., $\Omega(kk!, (k + 1)! - 1)$ have in common, we obtain

(2.5) $$D((k + 1)! - 1) = (k + 1)D(k! - 1) - \sum_{d=0}^{k-1} \lambda_{d,k}.$$

Let $A_k = D(k! - 1)/k!$ and $\varepsilon_k = \sum_{d=0}^{k-1} \lambda_{d,k}/(k + 1)!$, $k \ge k_0$. Then 2.5 becomes

$$A_{k+1} - A_k = -\varepsilon_k.$$

Therefore,

$$A_{k+1} - A_k = -\varepsilon_k$$
$$A_k - A_{k-1} = -\varepsilon_{k-1}$$
$$\vdots$$
$$A_{k_0} - A_k = -\varepsilon_{k_0}$$

so $A_{k+1} - A_{k_0} = -\sum_{j=k_0}^{k} \varepsilon_j$, i.e., $A_{k+1} = A_{k_0} - \sum_{j=k_0}^{k-1} \varepsilon_j$. Replacing $k+1$ by k, we obtain

(2.6) $$A_k = A_{k_0} - \sum_{j=k_0}^{k-1} \varepsilon_j.$$

Clearly, $1/k! \le A_k \le 1$ and $\sum_{j=k_0}^{k-1} \varepsilon_j = A_{k_0} - A_k \le A_{k_0} \le 1$. Thus, $\sum_{j=k_0}^{\infty} \varepsilon_j$ is a series of nonnegative terms bounded by A_{k_0}, hence is convergent. Let

(2.7) $$L = A_{k_0} - \sum_{j=k_0}^{\infty} \varepsilon_j.$$

Note that we have just shown that $0 \le L \le 1$. Then, 2.6 yields

(2.8) $$A_k = L + \sum_{j=k}^{\infty} \varepsilon_j, \; k \ge k_0.$$

Since $\sum_{j=k}^{\infty} \varepsilon_j = 0(1)$ as $k \to \infty$, we have

$$A_k = L + o(1).$$

Multiplying both sides of this equation by $k!$ and using the definition of the A_k, we obtain

(2.9) $$D(k! - 1) = Lk! + o(k!).$$

Using 2.3, 2.4, 2.9, and the definition of the λ's and the ε's, we have

$$\begin{aligned} D(dk! - 1) &= \sum_{c=0}^{d-1} D(ck!, (c+1)k! - 1) - \sum_{c=0}^{d-2} \lambda_{c,k} \\ &= \sum_{c=0}^{d-1} (Lk! + o(k!)) + o((k+1)!\varepsilon_k) \\ &= dk!L + o((k+1)!) + o((k+1)!), \end{aligned}$$

i.e.,

(2.10) $$D(dk! - 1) = dk!L + o((k+1)!).$$

Now let $n = \sum_{j=0}^{k} d_j j!$ be any nonnegative integer. Then $D(n) = D(d_k k! - 1) + D(d_k k!, d_k k! + d_{k-1}(k-1)! + \cdots) - Q$, where Q is the number of elements that the sets $\Omega(0, d_k k! - 1)$ and $\Omega(d_k k!, d_k k! + d_{k-1}(k-1)! + \cdots)$ have in common. Hence, if n is sufficiently large, then, by using 2.2, 2.10, and the definition of the λ's, we obtain

$$\begin{aligned} D(n) &= d_k k!L + D(d_{k-1}(k-1)! + \cdots) + o((k+1)!) + o((k+1)!) \\ &= d_k k!L + D(d_{k-1}(k-1)! + \cdots) + o((k+1)!). \end{aligned}$$

Applying the same type of reasoning yields

$$D(d_{k-1}(k-1)! + \cdots) = d_{k-1}(k-1)! + o(k!)$$
$$= d_{k-1}(k-1)!L + o((k+1)!).$$

Continuing in this manner, we obtain

$$D(n) = L\left(n - \sum_{j=1}^{k_0-1} d_j j!\right) + D\left(\sum_{j=1}^{k_0-1} d_j j!\right) + (k - k_0),$$

errors of size $o((k+1)!)$.

Therefore,

$$D(n) = L\left(n - \sum_{j=1}^{k_0-1} d_j j!\right) + D\left(\sum_{j=1}^{k_0-1} d_j j!\right) + o(k!),$$

so

$$D(n)/n = L - L \cdot o(1) + o(1) + o(1),$$

which implies that the density of $\mathcal{R}$ is L, so the proof is complete.

Our next result is an immediate consequence of Theorem 2.1.

Corollary 2.11: If $f(d, j) = f(d)$ depends only on d, where $f(0) = 0$ and $f(d) = o(j!)$ uniformly in j for all other "digits" d, then the density of $\mathcal{R}$ is L, where L is defined as in equation 2.7.

Corollary 2.12: We have $L < 1$ if and only if the function $T(n)$ is not one-one.

Proof: We have $L = A_{k_0} - \sum_{j=k}^{\infty} \varepsilon_j = A_k - \sum_{j=k_0}^{\infty} \varepsilon_j$, for all $k \geq k_0$, where k_0 is defined as in Lemma 2.4. Therefore, $L \leq A_k$ if $k \geq k$. if $T(x) = T(y)$, $x \neq y$, and k is such that $k \geq k_0$ and $x \leq k! - 1$, $y \leq k! - 1$; then, since

$$A_k = D(k! - 1)/k!,$$

it follows that $L \leq A_k \leq 1$. If T is one-one, then it follows from the definition of the A's and the ε's that $A_k = 1$ and $\varepsilon_k = 0$ for all k, so $L = 1$.

It seems to be true, although possibly difficult to prove, that $L < 1$ if each $f(d, j) = f(d)$ depends only on d and f satisfies the hypotheses of Theoren 2.1. It also seems to be the case that we should always have $L > 0$ under these hypotheses; this result again will be left to conjecture.

3. EXISTENCE OF THE DENSITY WHEN $f(d, j) = 0(j!/j^2 \log^2 j)$

The main drawback to Theorem 2.1 is the condition $f(0, j) = 0$. If we assume that $f(d, j) = 0(j!)$ uniformly in j for all "digits" d, it seems to be difficult to find a workable relationship between the quantities A_k, but on the other hand, it also seems to be difficult to find an example of an image set $\mathcal{R}$ which does not have density under this assumption. However, we do have the following result.

Theorem 3.1: If $f(d, j) = 0(j!/j^2 \log^2 j)$ uniformly in j, then the density of $\mathcal{R}$ exists.

Proof: Let D and Ω be as before. If $n = \sum_{j=1}^{k} d\ j!$, then $S(n) = \sum_{j=1}^{k} 0(j!/j^2 \log^2 j) = 0(k!/k^2 \log^2 k)$.

Suppose that $r \leq s \leq t (r < t)$ and $s < (k+1)!$; then,

$$D(r, t) = D(r, s) + D(s+1, t) - |\Omega(r, s) \cap \Omega(s+1, t)|.$$

Since $S(n) = 0(k!/k^2 \log^2 k)$, we have

$$D(r, t) = D(r, s) + D(s+1, t) + 0(k!/k^2 \log^2 k). \tag{3.2}$$

In particular, if $r = 0$, $s = (k-1)! - 1$, and $t = k! - 1$, we obtain

$$D(k! - 1) = D(0, (k-1)! - 1) + D((k-1)!, k! - 1))$$
$$+ 0((k-1)!/(k-1)^2 \log^2 (k-1)).$$

Applying the same reasoning to compute the quantities $D(0, j! - 1)$, $2 \leq j \leq k - 1$, we see that

$$\begin{aligned} D(k! - 1) = D(0) &+ D(1!, 2! - 1) + D(2!, 3! - 1) + \cdots \\ &+ D((k-1)!, k! - 1) \\ &+ 0((k-1)!/(k-1)^2 \log^2 (k-1)) \\ &+ 0((k-2)!/(k-2)^2 \log^2 (k-2)) + \cdots \end{aligned}$$

so we finally obtain

$$D(k! - 1) = D(0) + \sum_{q=1}^{k-1} D(q!, (q+1)! - 1) + 0(k!/k^2 \log^2 k). \tag{3.3}$$

Now, by 3.2, we have

$$D(dk!, (d+1)k! - 1) = D(dk!, dk!) + D(dk! + 1, (dk+ 1)! - 1)$$
$$+ 0(k!/k^2 \log^2 k)$$

and by repeated application of 3.2, we obtain

$$D(dk!, (d+1)k! - 1) = D(dk!, dk!) + D(dk! + 1, dk! + \ ! - 1)$$
$$+ \cdots + D(dk! + (k-1)!, (d+1)k! - 1) + k$$
errors of size $0(k!/k^2 \log^2 k)$,

i.e.,

$$D(dk!, (d+1)k! - 1) = D(dk!, dk!) + \sum_{q=1}^{k-1} D(dk! + q!, dk! + (q+1)! - 1) + 0(k!/k \log^2 k). \tag{3.4}$$

Since all integers x which satisfy $dk! + q! \leq x \leq dk! + (q+1)! - 1$ have the same number of leading zeros, we have

$$D(dk! + q!, dk! + (q+1)! - 1) = D(q!, (q+1)! - 1),$$
$$1 \leq q \leq k - 1$$

(cf. the argument used to prove 2.2).

Using this fact, 3.4 becomes

$$D(dk!, (d+1)\ ! - 1) = D(0) + \sum_{q=1}^{k-1} D(q!, (q+1)! - 1) + 0(k!/k \log^2 k) \tag{3.5}$$

and 3.3 and 3.5 imply that

$$D(dk!, (d+1)k! - 1) = D(k! - 1) + 0(k!/k \log^2 k). \tag{3.6}$$

Now, using 3.6, we obtain

$$\begin{aligned} D((k+1)! - 1) &= D(k! - 1) + D(k!, (k+1)! - 1 + O(k!/k^2 \log^2 k) \\ &= D(k! - 1) + D(k!, 2k! - 1) + D(2k!, (k+1)! - 1) \\ &\qquad + O(k!/k^2 \log^2 k) + O(k!/k^2 \log^2 k) \\ &= 2D(k! - 1) + D(2k!, (k+1)! - 1) + O(k!/k \log^2 k). \end{aligned}$$

By repeated application of 3.6, we finally obtain

$$D((k+1)! - 1) = (k+1)D(k! - 1) + k + 1,$$

errors of size $O(k!/k \log^2 k)$;

thus,

(3.7) $$D((k+1)! - 1) = (k+1)D(k! - 1) + O((k+1)!/k \log^2 k).$$

Define $A_k = D(k! - 1)/k!$. Then 3.7 becomes

$$(k+1)!A_{k+1} - (k+1)!A_k = O((k+1)!/k \log^2 k);$$

and by telescoping, we see that

$$A_{k+1} = A_0 + \sum_{j=1}^{k} O(1/j \log^2 j).$$

It is not difficult to verify that $\sum_{j=1}^{k} O(1/j \log^2 j) = O(1/\log^2 k)$. Therefore, using the above equation, we may conclude that there exists a constant L such that

(3.8) $$A_k = L + O(1/\log k).$$

Now let $n = \sum_{j=1}^{m} d_{k_j} k!_j$ be any nonnegative integer, where each $d_{k_j} \neq 0$. Then

$$D(n) = D(d_{k_m} k!_m - 1) + D(d_{k_m} k! + d_{m-1} k!_{m-1} + \cdots) + O(k!_m / k_m^2 \log^2 k_m).$$

By the same type of reasoning employed to get 3.4 and 3.7, we see that

$$D(d_{k_m} k!_m - 1) = d_{k_m} D(k!_m - 1) + O(k!_m/k \log^2 k_m) + D(d_{k_m} k!_m, d_{k_m} k!_m + \cdots).$$

Since $d_{k_m} \neq 0$ for any j, we have

$$D\left(d_{k_m} k!_m, \sum_{j=1}^{m} d_{k_j} k!_j\right) = D\left(\sum_{j=1}^{m-1} d_{k_j} k!_j\right).$$

Therefore,

$$D(n) = d_{k_m} k!_m (L + O(1/\log k_m)) + O(k!_m / k_m \log^2 k_m) + D\left(\sum_{j=1}^{m-1} d_{k_j} k!_j\right).$$

Continuing in this manner yields

$$D(n) = nL + O(k!_m / k_m \log^2 k_m) + \sum_{j=1}^{k_m} O(j!/\log j).$$

Hence, $$D(n) = nL + O(k!_m / \log k_m),$$

so $$D(n)/n = L + O(1/\log k_m) = L + o(1),$$

which proves that the density of $\mathcal{R}$ is L.

Remark 1: Theorem 3.1 has the drawback that the computability of the density has been lost.

Remark 2: If we assume that $f(d, j) = o(j!)$ uniformly in j, then there exists an image set $\mathcal{R}$ which does not have density. For example, let $f(d, j) = 0$ when j is even and $f(d, j) = j!$ when j is odd. Then,

$$T\left(k! + \sum_{j=1}^{k-1} d_j j!\right) = k! + \sum_{j=1}^{k-1} d_j j! + k! + (k - 2) + \cdots + 1! \geq 2k!$$

if k is odd, and

$$T\left(k! + \sum_{j=1}^{k-1} d_j j!\right) = k! + \sum_{j=1}^{k-1} d_j j! + (k - 1)! + (k - 3)! + \cdots + 1!$$

if k is even. Therefore, the number of integers between $k!$ and $2k!$ that belong to $\mathcal{R}$ if k is odd is at most $1 + (k - 2)! + (k - 4)! + \cdots + 1$, and the number of integers between $k!$ and $2k!$ that belong to $\mathcal{R}$ if k is even is at $k! - (k - 1)! - (k - 3)! - \cdots - 1!$. Hence, if we let δ and Δ denote the lower and upper density of $\mathcal{R}$, respectively, we see that

$$\delta \leq 0 + o(1) \quad \text{and} \quad \Delta \geq 1 + o(1),$$

so $\delta = 0$ and $\Delta = 1$.

It is also interesting to note that, if we let $f(d, j) = o(j!)$ uniformly in j, there do exist image sets $\mathcal{R}$ of density 0. For example, if $f(d, j) = 0$ when $d \neq 1$ or $j = 1$ and $f(d, j) = 2j!$ if $d = 1$ and $j > 1$, then no member of (except 1) has the "digit" 1 anywhere in its factorial representation, and the set

$$\left\{ n \middle| n = \sum_{j=1}^{k} d_j j!,\ d_j \neq 1,\ 1 \leq j \leq k \right\} \tag{3.8}$$

is easily seen to be the set of density 0.

Our next result is an immediate corollary of Theorem 3.1.

Corollary 3.9: If $f(d, j) = f(d)$ depends only on d and

$$f(d) = 0(j!/j^2 \log^2 j)$$

uniformly in j, then the density of $\mathcal{R}$ exists.

Finally, just as in [2] and [3], we wish to consider the special case that arises when we assume that $f(d, j) = f(d) = d$ for all "digits" d [so that $T(n)$ is the function n + the sum of the "digits" of n]. Clearly, $f(d)$ satisfies the assumptions of Corollary 2.11, so we know that the density of $\mathcal{R}$ is L, where L is defined as in 2.7. In this case, it is easy to verify that $k_0 = 0$ and that the value of $\lambda_{d,k}$ does not depend on d. Let us therefore set $\lambda_{d,k} = \lambda_k$, $0 \leq d \leq k$. In the following table, we give the values of λ_k and ε_k to the nearest 6 decimal places; it appears to be difficult to develop an algorithm to calculate the λ_k in general.

Using this table together with Taylor's formula and Lagrange's form for the remainder, we obtain the following result.

Theorem 3.10: When $T(n)$ is the function n + the sum of the "digits" of n, the density of $\mathcal{R}$ is 0.879888. The error made using this figure is less than $e/2 \cdot 9!$. Therefore, $\mathcal{R}$ has positive density in this case.

The Values of λ_k and ε_k, $1 \le k \le 10$

k	λ_k	ε_k
1	0	0
2	0	0
3	0	0
4	2	0.066667
5	6	0.041667
6	8	0.008929
7	14	0.002401
8	17	0.000375
9	26	0.000064
10	39	0.000009

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EVALUATION OF SUMS OF CONVOLVED POWERS USING STIRLING AND EULERIAN NUMBERS

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ABSTRACT

It is shown here how the method of generating functions leads quickly to compact formulas for sums of the type

$$S(i,j;n) = \sum_{0 \le k \le n} k^i (n-k)^j$$

using Stirling numbers of the second kind and also using Eulerian numbers. The formulas are, for the most part, much simpler than corresponding results using Bernoulli numbers.

1. INTRODUCTION

Neuman and Schonbach [9] have obtained a formula for the series of convolved powers

$$S(i,j;n) = \sum_{k=0}^{n} k^i (n-k)^j \tag{1.1}$$