DIVISIBILITY PROPERTIES OF POLYNOMIALS IN PASCAL'S TRIANGLE

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Divisibility properties of the Fibonacci sequence $\{F_n\}$ are well known, including the property of greatest common divisors,

$$(F_m, F_n) = F_{(m,n)}.$$

Here the derivation of the greatest common divisor of a sequence pair is extended to the Fibonacci polynomials, the Morgan-Voyce polynomials, the Chebyshev polynomials, and more general polynomials from a problem of Schechter [1]. Moreover, all of these polynomials have coefficients which lie along rising diagonals of Pascal's triangle, and all of these polynomials satisfy $(u_m(x), u_n(x)) = u_{(m,n)}(x)$ with suitable adjustment of subscripts.

1. INTRODUCTION

The Morgan-Voyce polynomials in [2], [3], and [4] are defined by

$$B_0(x) = 1,\ B_1(x) = x + 2;\ b_0(x) = 1,\ b_1(x) = x + 1,$$

and

$$B_n(x) = b_{n-1}(x) + (1 + x)B_{n-1}(x),$$

$$b_n(x) = xB_{n-1}(x) + b_{n-1}(x), \tag{1.1}$$

$$B_n(x) = B_{n-1}(x) + b_n(x).$$

It is easy to show that $B_{-1}(x) = 0$, and $b_{-1}(x) = 1$. These mixed recurrences could be solved for pure recurrences as each separately satisfies

$$u_{n+2}(x) = (x + 2)u_{n+1}(x) - u_n(x), \tag{1.2}$$

with $u_0 = 1$ and $u_1 = x + 2$, and $u_0 = 1$ and $u_1 = x + 1$, respectively.

If one lists these polynomials,

$$\begin{aligned}
b_0(x) &= 1\\
B_0(x) &= 1\\
b_1(x) &= x + 1\\
B_1(x) &= x + 2\\
b_2(x) &= x^2 + 3x + 1\\
B_2(x) &= x^2 + 4x + 3\\
b_3(x) &= x^3 + 5x^2 + 6x + 1\\
B_3(x) &= x^3 + 6x^2 + 10x + 4\\
&\vdots
\end{aligned}$$

Clearly, we see that the coefficients of this double sequence lie along the rising diagonals of Pascal's triangle.

The Fibonacci polynomials are

$$f_0(x) = 0,\ f_1(x) = 1,\ f_{n+2}(x) = xf_{n+1}(x) + f_n(x), \tag{1.3}$$

and we list the first few of these polynomials:

$$\begin{aligned}
f_1(x) &= 1\\
f_2(x) &= x\\
f_3(x) &= x^2 + 1\\
f_4(x) &= x^3 + 2x
\end{aligned}$$

$$\begin{aligned} f_5(x) &= x^4 + 3x^2 + 1 \\ f_6(x) &= x^5 + 4x^2 + 3x \\ f_7(x) &= x^6 + 5x^4 + 6x^2 + 1 \\ f_8(x) &= x^7 + 6x^5 + 10x^3 + 4x \\ &\vdots \end{aligned}$$

Once again, we see that the coefficients lie along the rising diagonals of Pascal's triangle.

It can be shown that [3], [4]

$$(1.4)\qquad \begin{aligned} b_n(x^2) &= f_{2n+1}(x) \\ xB_n(x^2) &= f_{2n+2}(x), \end{aligned}$$

and the fact that coefficients lie on the rising diagonals of Pascal's triangle follows from that property for the Fibonacci polynomials. The Fibonacci polynomials obey

$$(1.5)\qquad f_{n+4}(x) = (x^2 + 2)f_{n+2}(x) - f_n(x),$$

which agrees with (1.2) when x is replaced by x^2 throughout.

Next, we are interested in finding the greatest common divisor of a pair of Fibonacci polynomials.

Theorem 1.1: For Fibonacci polynomials,

$$(f_m(x),\ f_n(x)) = f_{(m,n)}(x).$$

Proof: Rewrite the recursion (1.3) for the Fibonacci polynomials,

$$f_{m+1}(x) - xf_m(x) = f_{m-1}(x),$$

and set $(f_m(x),\ f_{m+1}(x)) = d(x)$. Then, since $d(x)|f_m(x)$ and $d(x)|f_{m+1}(x)$, we must have $d(x)|f_{m-1}(x)$. In turn, $f_m(x) - xf_{m-1}(x) = f_{m-2}(x)$ implies that $d(x)|f_{m-2}(x)$, and, continuing, finally $d(x)|f_1(x) = 1$. Therefore, $d(x) = 1$, and Theorem 1.1 holds for $n = m + 1$, or,

$$(1.6)\qquad (f_m(x),\ f_{m+1}(x)) = 1.$$

From [5], we also have

$$(1.7)\qquad f_{p+r}(x) = f_{p-1}(x)f_r(x) + f_p(x)f_{r+1}(x),$$

and

$$(1.8)\qquad f_m(x)|f_n(x) \text{ if and only if } m|n.$$

Next, let $c = (m,\ n)$, and let $d(x) = (f_m(x),\ f_n(x))$. Since $c|m$ and $c|n$, by (1.8), $f_c(x)|f_m(x)$ and $f_c(x)|f_n(x)$ implies that $f_c(x)|d(x)$. Since $c = (m, n)$, by the Euclidean algorithm, there exist integers a and b such that $c = am + bn$. Since $c \le m$, $m, n > 0$, $a \le 0$ or $b \le 0$. Suppose $a \le 0$ and let $k = -a$. Then $bn = c + km$ applied to (1.7) gives

$$f_{bn}(x) = f_{c+km}(x) = f_{c-1}(x)f_{km}(x) + f_c(x)f_{km+1}(x).$$

By (1.8), $f_n(x)|f_{bn}(x)$ and $f_m(x)|f_{km}(x)$, and since $d(x)|f_n(x)$ and $d(x)|f_m(x)$, we have $d(x)|f_c(x)f_{km+1}(x)$. But $(f_{km}(x),\ f_{km+1}(x)) = 1$ by (1.6), which implies that $(d(x), f_{km+1}(x)) = 1$, and $d(x) | f_c(x)$. Also, since $f_c(x) | d(x)$, $d(x) = f_c(x)$, or $(f_m(x), f_n(x)) = f_{(m,n)}(x)$, concluding the proof, which is similar to that by Michael [6] for Fibonacci numbers. Also see [7] and [8].

2. POLYNOMIALS FROM A PROBLEM BY SCHECHTER

Next, we consider some polynomials arising from a problem by Schechter [1] and their relationships to the Fibonacci polynomials and the Morgan-Voyce polynomials. Consider the sequence defined by $S_1 = 1$, $S_2 = m$, and

$$(2.1)\qquad \begin{cases} S_k = mS_{k-1} + S_{k-2}, & k \text{ even,} \\ S_k = nS_{k-1} + S_{k-2}, & k \text{ odd.} \end{cases}$$

We now list the first few polynomials in m and n, and compare to the Morgan-Voyce polynomials.

$$\begin{aligned}
S_1(m, n) & & &= b_0(mn) \\
S_2(m, n) &= m & &= mB_0(mn) \\
S_3(m, n) &= mn + 1 & &= b_1(mn) \\
S_4(m, n) &= m(mn + 2) & &= mB_1(mn) \\
S_5(m, n) &= (mn)^2 + 3mn + 1 & &= b_2(mn) \\
S_6(m, n) &= m[(mn)^2 + 4mn + 3] & &= mB_2(mn)
\end{aligned}$$

Thus, it appears that

$$(2.2)\qquad \begin{cases} S_{2k+2}(m, n) = mB_k(mn), \\ S_{2k+1}(m, n) = b_k(mn). \end{cases}$$

Now, from (1.4), we have $mnB_k(m^2n^2) = f_{2k+2}(mn)$; thus,

$$(2.3)\qquad S_{2k+2}(m^2, n^2) = m^2B_k(m^2n^2) = \frac{m}{n}f_{2k+2}(mn).$$

For example, $S_4(m^2, n^2) = m^2(m^2n^2 + 2)$, $B_1(m^2n^2) = m^2n^2 + 2$, and $f_4(mn) = (mn)^3 + 2mn$, and we see that

$$\begin{aligned}
S_4(m^2, n^2) &= m^2B_1(m^2n^2) = \frac{m}{n}(mn)(m^2n^2 + 2) \\
&= \frac{m}{n}(m^3n^3 + 2mn) = \frac{m}{n}f_4(mn).
\end{aligned}$$

Next, we state and prove a matrix theorem in order to derive further results for the polynomials $S_k(m, n)$.

Theorem 2.1: Let $A = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}$. Then,

$$(AB)^k = \begin{pmatrix} b_k(xy) & xB_{k-1}(xy) \\ yB_{k-1}(xy) & b_{k-1}(xy) \end{pmatrix},$$

where $b_k(x)$ and $B_k(x)$ are the Morgan-Voyce polynomials.

Proof:

$$(AB)^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_0(xy) & xB_{-1}(xy) \\ yB_{-1}(xy) & b_{-1}(xy) \end{pmatrix}$$

$$(AB)^1 = \begin{pmatrix} xy+1 & x \\ y & 1 \end{pmatrix} = \begin{pmatrix} b_1(xy) & xB_0(xy) \\ yB_0(xy) & b_0(xy) \end{pmatrix}$$

Assume that $(AB)^k$ has the form of the theorem. Then,

$$(AB)(AB)^k = \begin{pmatrix} xy+1 & x \\ y & 1 \end{pmatrix} \cdot \begin{pmatrix} b_k(xy) & xB_{k-1}(xy) \\ yB_{k-1}(xy) & b_{k-1}(xy) \end{pmatrix}$$

$$= \begin{pmatrix} xyb_k(xy) + xyB_{k-1}(xy) + b_k(xy) & x[(xy+1)B_{k-1}(xy) + b_{k-1}(xy)] \\ yb_k(xy) + yB_{k-1}(xy) & xyB_{k-1}(xy) + b_{k-1}(xy) \end{pmatrix}$$

$$= \begin{pmatrix} b_{k-1}(xy) & xB_k(xy) \\ yB_k(xy) & b_k(xy) \end{pmatrix},$$

by applying the mixed recurrences of (1.1), completing a proof by induction.

Now, returning to the matrices of Theorem 2.1, since the determinant of AB is 1, it follows that

$$b_k(xy)b_{k-1}(xy) - xyB_{k-1}^2 = 1. \tag{2.4}$$

Returning to the polynomials $S_k(m, n)$, we have also that

$$(AB)^k = \begin{pmatrix} S_{2k+1} & \frac{n}{m}S_{2k} \\ S_{2k} & S_{2k-1} \end{pmatrix},$$

so that, taking determinants,

$$S_{2k-1}S_{2k+1} - \frac{n}{m}S_{2k}^2 = 1. \tag{2.5}$$

The polynomials $S_k(m, n)$ are related to the Morgan-Voyce polynomials by

$$\begin{cases} S_{2k+1}(m, n) = b_k(mn), \\ \frac{n}{m}S_{2k}(m, n) = nB_{k-1}(mn), \\ S_{2k}(m, n) = mB_{k-1}(mn). \end{cases} \tag{2.6}$$

Since the polynomials $S_k(m, n)$, the Morgan-Voyce polynomials, and the Fibonacci polynomials are interrelated by (1.4) and (2.3), which can be rewritten as

$$\begin{cases} S_{2k+1}(m, n) = f_{2k+1}(\sqrt{mn}), \\ S_{2k}(m, n) = \frac{m}{\sqrt{mn}}f_{2k}(\sqrt{mn}), \end{cases} \tag{2.7}$$

and since the coefficients of the Fibonacci polynomials lie along the rising diagonals of Pascal's triangle, we can write the following theorem.

Theorem 2.2: The coefficients of $f_k(x)$, $b_k(x)$, $B_k(x)$, and $S_k(m, n)$ are all coefficients which lie along the rising diagonals of Pascal's triangle.

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Using the relationships of §2, we can expand upon Theorem 1.1 to write a greatest common divisor property for Morgan-Voyce polynomials.

Theorem 3.1: For the Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$,

(i) $(B_m(x), B_n(x)) = B_{(m+1,\ n+1)-1}(x)$,

(ii) $(b_m(x), b_n(x)) = b_{((2m+1,\ 2n+1)-1)/2}(x)$,

(iii) $(B_m(x), b_n(x)) = b_{((2m+2,\ 2n+1)-1)/2}(x)$.

Proof:

(i)
$$x(B_m(x^2), B_n(x^2)) = (f_{2m+2}(x), f_{2n+2}(x)) = f_{2(m+1,\ n+1)}(x) = xB_{(m+1,\ n+1)-1}(x^2)$$

by applying (1.4), Theorem 1.1, and returning to (1.4). For $x \neq 0$, (i) is immediate by replacing x^2 with x after dividing both sides by x. If $x = 0$, $B_n = n + 1$, making (i) become $(m + 1, n + 1) = (m + 1, n + 1) - 1 + 1$.

Applying (1.4) and Theorem 1.1 to (ii),

$$(b_m(x^2), b_n(x^2)) = (f_{2m+1}(x), f_{2n+1}(x)) = f_{(2m+1,\ 2n+1)}(x) = f_{2k+1}(x)$$

since the greatest common divisor of $2m + 1$ and $2n + 1$ is odd. Thus,

$$(b_m(x^2), b_n(x^2)) = b_k(x^2)$$

by (1.4), where $2k + 1 = (2m + 1, 2n + 1)$, so that

$$k = ((2m + 1, 2n + 1) - 1)/2.$$

Replacing x^2 by x yields (ii).

Finally, we observe that $b_n(0) = 1$, so that $x \nmid b_n(x)$, and again use (1.4) and Theorem 1.1:

$$(B_m(x^2), b_n(x^2)) = (xB_m(x^2), b_n(x^2)) = (f_{2m+2}(x), f_{2n+1}(x)) = f_{(2m+2,\ 2n+1)}(x).$$

Next, set $(2m + 2, 2n + 1) = 2k + 1$, since it must be odd, and

$$(B_m(x^2), b_n(x^2)) = f_{2k+1}(x) = b_k(x^2)$$

where

$$k = ((2m + 2, 2n + 1) - 1)/2.$$

Replacing x^2 by x establishes (iii), finishing the proof of Theorem 3.1.

Returning to the polynomials $S_k(m, n)$, and using (2.7) with Theorem 1.1, gives us

Theorem 3.2: $(S_i(m, n), S_j(m, n)) = S_{(i,\ j)}(m, n)$.

Proof: If i and j are both odd, (2,7) and Theorem 1.1 give the above result immediately. If i and j are both even,

$$\begin{aligned}(S_i(m, n), S_j(m, n)) &= (S_{2k}(m, n), S_{2h}(m, n))\\ &= \left(\frac{m}{\sqrt{mn}} f_{2k}(\sqrt{mn}), \frac{m}{\sqrt{mn}} f_{2h}(\sqrt{mn})\right)\\ &= \frac{m}{\sqrt{mn}}(f_{2k}(\sqrt{mn}), f_{2h}(\sqrt{mn})) = \frac{m}{\sqrt{mn}} f_{2(k,h)}(\sqrt{mn})\\ &= S_{2(k,h)}(m, n) = S_{(2k,2h)}(m, n) = S_{(i,j)}(m, n).\end{aligned}$$

If i is odd and j is even, since $S_{2k+1}(m, n)$ always ends in the constant 1 so that $\sqrt{mn} \nmid S_{2k+1}(m, n)$, and since $f_{2k+1}(x)$ also ends in 1,

$$\begin{aligned}(S_i(m, n), S_j(m, n)) &= (S_{2k+1}(m, n), S_{2h}(m, n))\\ &= (S_{2k+1}(m, n), \sqrt{mn}S_{2h}(m, n))\\ &= (f_{2k+1}(\sqrt{mn}), mf_{2h}(\sqrt{mn})) = (f_{2k+1}(\sqrt{mn}), f_{2h}(\sqrt{mn}))\\ &= f_{(2k+1, 2h)}(\sqrt{mn}) = S_{(2k+1, 2h)}(m, n) = S_{(i,j)}(m, n),\end{aligned}$$

where we can again use (2.7) because $(2k+1, 2h)$ is odd, concluding the proof of Theorem 3.2.

We quickly have divisibility properties for the polynomials $S_k(m, n)$.

Theorem 3.3: $S_i(m, n) | S_j(m, n)$ if and only if $i | j$.

Proof: If $i | j$, then $(i, j) = i$, and $S_i(m, n) | S_j(m, n)$ by Theorem 3.2. If $S_i(m, n) | S_j(m, n)$ with $i \nmid j$, then $f_i(x) | f_j(x)$ where $i \nmid j$, a contradiction of (1.8).

From all of this, we can also write divisibility properties for Morgan-Voyce polynomials.

Theorem 3.4: For the Morgan-Voyce polynomials,

$B_m(x) | B_n(x)$ if and only if $(m + 1) | (n + 1)$;

$b_m(x) | b_n(x)$ if and only if $(2m + 1) | (2n + 1)$;

$b_m(x) | B_n(x)$ if and only if $(2m + 1) | (n + 1)$.

Proof: $B_m(x) | B_n(x)$ if and only if $(B_m(x), B_n(x)) = B_m(x)$, but

$$(B_m(x), B_n(x)) = B_{(m+1, n+1)-1}(x)$$

by Theorem 3.1. Setting the subscripts equal, $m = (m+1, n+1) - 1$, or, $m + 1 = (m + 1, n + 1)$, which forces $(m + 1) | (n + 1)$. The case for $b_m(x)$ and $b_n(x)$ is entirely similar.

In the case of $b_m(x)$ and $B_n(x)$, $B_n(x)$ cannot divide $b_m(x)$ for $n > 0$ because $b_m(x)$ always ends in the constant 1, while the constant for $B_n(x)$ is greater than 1, $n > 0$. Since $b_m(x) | B_n(x)$ if and only if

$$(b_m(x), B_n(x)) = b_m(x),$$

and since

$$(b_m(x), B_n(x)) = b_{((2n+2, 2m+1)-1)/2}(x)$$

by carefully rearranging (iii) in Theorem 3.1, equating the subscripts leads to

$$m = ((2n + 2, 2m + 1) - 1)/2,$$

or

$$2m + 1 = (2m + 1, 2n + 2).$$

Thus, $(2m + 1)|(2n + 2)$, but since $(2m + 1)$ is odd, we must have

$$(2m + 1)|(n + 1),$$

concluding the proof.

Returning to the greatest common divisor property of the Fibonacci polynomials, $(f_m(x), f_n(x)) = f_{(m,n)}(x)$, we make some observations from Theorem 3.1(i) regarding the Morgan-Voyce polynomials $B_n(x)$. From

$$(B_n(x), B_m(x)) = B_{(n+1, m+1)-1}(x),$$

it would follow that if $B_n^*(x) = B_{n-1}(x)$ and $B_m^*(x) = B_{m-1}(x)$, then

(3.1) $$(B_n^*(x), B_m^*(x)) = B_{(n,m)}^*$$

which sequence $\{B_n^*(x)\} = \{0, 1, x + 2, \ldots\}$ obeys

(3.2) $$B_n^*(x) = (x + 2)B_{n-1}^*(x) - B_{n-2}^*(x)$$

and is in fact the Fibonacci polynomial, so to speak, for the auxiliary polynomial $\lambda^2 - (x + 2)\lambda + 1 = 0$, since

$$B_n^*(x) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$$

where λ_1 and λ_2 are the roots. But (3.2) can also be expressed as

$$u_n = xu_{n-1} - u_{n-2}$$

where x is replaced by $(x + 2)$. Thus one set of polynomials with coefficients on diagonals of Pascal's triangle transforms into another set with the same property.

This property of transforming one set of polynomials whose coefficients are on diagonals of Pascal's triangle to another set of polynomials with coefficients also on diagonals of Pascal's triangle is shared by the Chebyshev polynomials $\{T_n(x)\}$ [9] of the first kind, defined by $T_0(x) = 1$, $T_1(x) = x$, and

(3.3) $$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

since

(3.4) $$T_n(T_m(x)) = T_m(T_n(x)) = T_{mn}(x).$$

The property (3.4) is easy to prove from the Binet form associated with the auxiliary polynomial

(3.5) $$\lambda^2 - 2x\lambda + 1 = 0,$$

with roots λ_1 and λ_2.

The Chebyshev polynomials $\{U_n(x)\}$ of the second kind are $U_0(x) = 1$, and $U_1(x) = 2x$,

(3.6) $$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

First, to establish (3.4), we prove by induction that

(3.7) $$\begin{cases} \lambda_1^n = T_n(x) + \sqrt{(x^2 - 1)}U_{n-1}(x), \\ \lambda_2^n = T_n(x) - \sqrt{(x^2 - 1)}U_{n-1}(x). \end{cases}$$

We prove only one part, since the second part is entirely similar. Since, $U_{-1}(x) = 0$, and $T_0(x) = 1$, $\lambda_1^n = T_n(x) + \sqrt{(x^2 - 1)}U_{n-1}(x)$ for $n = 0$. Assume that $\lambda_1^k = T_k(x) + \sqrt{(x^2 - 1)}U_{k-1}(x)$ and $\lambda_1^{k+1} = T_{k+1}(x) + \sqrt{(x^2 - 1)}\, U\ (x)$. Then, by (3.5),

$$\lambda_1^{k+2} = 2x\lambda_1^{k+1} - \lambda_1^k = (2xT_{k+1}(x) - T_k(x)) + \sqrt{(x^2 - 1)}\,(2xU_{k+1}(x) - U_k(x))$$
$$= T_{k+2}(x) + \sqrt{(x^2 - 1)}U_{k+1}(x),$$

using (3.4) and (3.6), establishing the form of λ_1^n in (3.7) by mathematical induction.

Notice that, since $\lambda_1\lambda_2 = 1$, by multiplying the forms of λ_1^n and λ_2^n from (3.7), we can derive

(3.8) $$T_n^2(x) - 1 = (x^2 - 1)U_{n-1}^2(x).$$

Also, by adding in (3.7), we can establish

(3.9) $$T_n(x) = (\lambda_1^n + \lambda_2^n)/2.$$

Now, $\lambda_1(x) = x + \sqrt{x^2 - 1}$. Replace x by $T_m(x)$, and the root becomes

$$\lambda_1(T_m(x)) = T_m(x) + \sqrt{T_m^2(x) - 1},$$

satisfying the auxiliary polynomial (3.5), so that

$$\lambda_1^2(T_m(x)) - 2T_m(x)\lambda_1(T_m(x)) + 1 = 0.$$

That is,

$$T_m(x) = \frac{\lambda_1^2(T_m(x)) + 1}{2\lambda_1(T_m(x))} = [\lambda_1(T_m(x)) + 1/\lambda_1(T_m(x))]/2.$$

But $\lambda_1\lambda_2 = 1$, so

$$T_m(x) = [\lambda_1(T_m(x)) + \lambda_2(T_m(x))]/2.$$

Referring back to (3.9), we write

$$\lambda_1 = \lambda_1^m(T_m(x)) \quad \text{and} \quad \lambda_2^m = \lambda_2(T_m(x)).$$

Now,

$$T_{mn}(x) = [\lambda_1^{mn} + \lambda_2^{mn}]/2 = [(\lambda_1^m)^n + (\lambda_2^m)^n]/2 = [\lambda_1^n(T_m(x)) + \lambda_2^n(T_m(x))]/2,$$

so that $T_{mn}(x) = T_n(T_m(x))$ and similarly, $T_{mn}(x) = T_m(T_n(x))$, finishing the proof of (3.4).

Returning to divisibility properties, observe that the Chebyshev polynomials of the second kind are the polynomials with the Fibonacci-like property

$$U_{n-1}(x) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$$

where λ_1 and λ_2 are the roots of $\lambda^2 - 2x\lambda + 1 = 0$. We now list the first few polynomials and let

$$U_n^*(x) = U_{n-1}(x).$$

$$\begin{aligned}
U_{-1}(x) &= 0 &&= U_0^*(x)\\
U_0(x) &= 1 &&= U_1^*(x)\\
U_1(x) &= 2x &&= U_2^*(x)\\
U_2(x) &= 4x^2 - 1 &&= U_3^*(x)\\
U_3(x) &= 8x^3 - 4x = 4x(2x^2 - 1) &&= U_4^*(x)\\
U_4(x) &= 16x^4 - 12x^2 + 1 &&= U_5^*(x)
\end{aligned}$$

$$U_5(x) = 32x^5 - 32x^3 + 6x = 2x(8x^4 - 8x^2 + 3) = U_6^*(x)$$
$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1 = U_7^*(x)$$
$$\vdots \qquad\qquad \vdots$$

It would appear that

$$(3.10) \qquad U_m^*(x),\ U_n^*(x) = U_{(m,n)}^*(x).$$

That this is indeed the case can be established very simply. Since $U_n^*(x)$ satisfies

$$U_{n+1}^*(x) = 2xU_n^*(x) - U_{n-1}^*(x),$$

$\{U_n(x)\}$ is a special case of the polynomial sequence $\{U_n(x, y)\}$ defined by Hoggatt and Long [7] as

$$(3.11) \qquad U_{n+2}(x, y) = xU_{n+1}(x, y) + yU_n(x, y),$$

where $U_0(x, y) = 0$ and $U_1(x, y) = 1$. Note that $\{U_n^*(x)\}$ is the special case $x = 2x$ and $y = -1$. Since

$$(3.12) \qquad \big(U_m(x, y),\ U_n(x, y)\big) = U_{(m,n)}(x, y),$$

we see that (3.10) is immediate.

We summarize as

Theorem 3.4: By suitable shifting of subscripts in the original definitions, the Fibonacci Polynomials, the Morgan-Voyce polynomials $B_n(x)$, the Chebyshev polynomials $U_n(x)$, and the polynomials $S_k(m, n)$ all satisfy

$$(u_m,\ u_n) = u_{(m,n)}.$$

4. A MORE GENERAL POLYNOMIAL SEQUENCE

Define $S_k(a, b, c, d)$ by taking $S_1 = 1$, $S_2 = a$,

$$(4.1) \qquad \begin{cases} S_k = aS_{k-1} + bS_{k-2}, & k \text{ even}, \\ S_k = cS_{k-1} + dS_{k-2}, & k \text{ odd}. \end{cases}$$

Let $S_1^* = 1$, $S_2^* = c$, and define $S_k^*(a, b, c, d)$ by taking

$$(4.2) \qquad \begin{cases} S_k^* = cS_{k-1}^* + dS_{k-2}^*, & k \text{ even}, \\ S_k^* = aS_{k-1}^* + bS_{k-2}^*, & k \text{ odd}. \end{cases}$$

Let $K_0 = 0$, $K_1 = 1$, $K_n = (ac + b + d)K_{n-1} - bdK_{n-2}$.

Let $Q = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c & d \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ac + b & ad \\ c & d \end{pmatrix}$; then,

$$Q^k = \begin{pmatrix} S_{2k+1}^* & dS_{2k} \\ S_{2k}^* & dS_{2k-1} \end{pmatrix} = \begin{pmatrix} K_{k+1} - dK_k & daK_k \\ cK_k & d(K_k - bK_{k-1}) \end{pmatrix}$$

Now, $\{K_n\}$ is the "Fibonacci sequence,"

$$K_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

for the quadratic $\lambda^2 - (ac + b + d)^\lambda + bd = 0$, with roots λ_1, λ_2. Applying results [7] for $\{U_n(x, y)\}$ from (3.11) and (3.12) to $\{K_n\}$, we have immediately that

$$(K_m, K_n) = K_{(m,n)}.$$

To continue, we write the first few terms of $\{S_k(a, b, c, d)\}$.

$$S_1 = 1$$
$$S_2 = a$$
$$S_3 = ac + d$$
$$S_4 = a^2c + ad + ab$$
$$S_5 = a^2c^2 + 2acd + abc + d^2$$
$$S_6 = a^3c^2 + 2a^2cd + 2a^2bc + ad^2 + abd + ab^2$$
$$S_7 = a^3c^3 + 3a^2c^2d + 2a^2bc^2 + 3acd^2 + 2abcd + ab^2c + d^3$$

We consider some special cases. If $a = 0$, then $S_{2k+2} = 0$, and $S_{2k+1} = d^k$, $k \geq 0$. If $b = 0$, $S_{2k+2} = a(ac + d)^k$ and $S_{2k+1} = (ac + d)^k$, $k \geq 0$. If $c = 0$, then $S_{2k-1} = d^{k-1}$ and $S_{2k} = a[(d^k - b^k)/(d - b)]$, $k \geq 1$. If $d = 0$, then $S_{2k} = a(ac + b)^{k-1}$ and $S_{2k+1} = ac(ac + b)^{k-1}$, $k \geq 1$. The expansions of $S_k^*(a, b, c, d)$ are not very interesting, since they are the same as those of $S_k(a, b, c, d)$ with the roles of a and c exchanged.

The special case of $S_k(a, b, c, d)$ where $b = d$ proves fruitful. We list the first few terms of $\{S_k(a, b, c)\}$ below:

$$S_1 = 1$$
$$S_2 = a$$
$$S_3 = ac + b$$
$$S_4 = a^2c + 2ab$$
$$S_5 = a^2c^2 + 3abc + b^2$$
$$S_6 = a^3c^2 + 4a^2bc + 3ab^2 = a(ac + b)(ac + 3b) = S_2S_3(ac + 3b)$$

We are interested in the case $b = d$, or, taking $S_k(a, b, c)$ and $S_k^*(a, b, c)$, so that S_3 will divide S_6. It is not difficult to prove by induction that

(4.3) $$S_{2k+j} = S_{j+1}^*S_{2k} + bS_jS_{2k-1},$$

(4.4) $$S_{2k+1+j} = S_{j+1}S_{2k+1} + bS^*S_{2k}.$$

It is not hard to see that

(4.5) $$S_{2k+1} = S_{2k+1}^* \quad \text{and} \quad aS_{2k} = cS_{2k}^*.$$

We now prove $S_j | S_{jm}$ for j odd and m odd, or, $jm = 2k + 1$. From (4.4),

$$S_{j(m+1)} = S_{j+1}S_{jm} + bS_j^*S_{2k} = S_{j+1}S_{jm} + bS_jS_{2k},$$

since $S_j = S_j^*$ for j odd. So, if $S_j | S_j$ and $S_j | S_{jm}$, then $S_j | S_{j(m+1)}$ for j odd. Thus, for j and m both odd, we see that $S_j | S_{jm}$ for all odd m.

Next, suppose that j is odd and m is even; then, from (4.3),

$$S_{2m'j+j} = S_{j+1}^*S_{2m'j} + bS_jS_{2m'j-1}, \quad m = 2m'.$$

Now, if $S_j | S_j$ and $S_j | S_{2m'j}$, then $S_j | S_{(2m'+1)j} = S_{j(m+1)}$.

Next, let j be even;

$$S_{2k+2j} = S^*_{2j'+1} S_{2k} + bS_{2j'} S_{2k-1} \quad \text{and} \quad 2k = 2j'm.$$

Since $S_j | S_{2j'}$ and $S_j | S_{2j'm} = S_{2k}$, we have $S_j | S_{2j'm+2j'} = S_{j(m+1)}$. This completes the proof that if $i|j$, then $S_i | S_j$. Since, algebraically, $\{S_i\}$ are of increasing degree in the two variables a and c collectively, $S_j \nmid S_i$ for $i < j$. Last, using (4.3) and (4.4), it is now straightforward to show

Theorem 4.1: $S_j(a, b, c) | S_i(a, b, c)$ if and only if $j|i$.

We can also now prove

Theorem 4.2: $(S_i(a, b, c), S_j(a, b, c)) = S_{(i,j)}(a, b, c)$.

Proof: Let $P(x)$ be a monic polynomial of degree $r + s$ with integral coefficients with two factors $Q(x)$ and $R(x)$ of degree r and s, respectively. Then,

$$b^{r+s}P(x/b) = b^r Q(x/b) b^s R(x/b)$$

$$P^*(x, b) = Q^*(x, b) R^*(x, b).$$

In particular, if $P(x)$ is of degree p, $T(x)$ of degree t, $W(x)$ of degree w, and $(P(x), T(x)) = W(x)$, then

$$(b^p P(x/b), b^t T(x/b)) = b^w W(x/b).$$

For application to Theorem 4.2:

$$c^2 S_{2m}(a^2, b^2, c^2) = acb^{2m-1} f_{2m}(ac/b); \tag{4.6}$$

$$S_{2m+1}(a^2, b^2, c^2) = b^{2m} f_{2m+1}(ac/b). \tag{4.7}$$

Case 1: Both subscripts even.

$$\begin{aligned} &(c^2 S_{2m}(a^2, b^2, c^2), c^2 S_{2n}(a^2, b^2, c^2)) \\ &= (acb^{2m-1} f_{2m}(ac/b), acb^{2n-1} f_{2n}(ac/b)) \\ &= acb^{(2m, 2n)-1} f_{(2m, 2n)}(ac/b) \\ &= c^2 S_{(2m, 2n)}(a^2, b^2, c^2). \end{aligned}$$

Therefore,

$$(S_{2m}(a^2, b^2, c^2), S_{2n}(a^2, b^2, c^2)) = S_{(2m, 2n)}(a^2, b^2, c^2).$$

Case 2: Both subscripts odd.

$$\begin{aligned} &(S_{2m+1}(a^2, b^2, c^2), S_{2n+1}(a^2, b^2, c^2)) \\ &= (b^{2m} f_{2m+1}(ac/b), b^{2n} f_{2n+1}(ac/b)) \\ &= b^{(2m+1, 2n+1)-1} f_{(2m+1, 2n+1)}(ac/b) \\ &= S_{(2m+1, 2n+1)}(a^2, b^2, c^2). \end{aligned}$$

Case 3: One subscript odd, one subscript even.

$$\begin{aligned} &(c^2 S_{2m}(a^2, b^2, c^2), S_{2n+1}(a^2, b^2, c^2)) \\ &= (acb^{2m-1} f_{2m}(ac/b), b^{2n} f_{2n+1}(ac/b)) \\ &= b^{(2m, 2n+1)-1} f_{(2m, 2n+1)}(ac/b) \\ &= S_{(2m, 2n+1)}(a^2, b^2, c^2), \end{aligned}$$

since $(ac, b) = 1$. Also, since $(c^2, S_{2n+1}) = 1$,

$$\begin{aligned}&(c^2S_{2m}(a^2, b^2, c^2), S_{2n+1}(a^2, b^2, c^2))\\ &= (S_{2m}(a^2, b^2, c^2), S_{2n+1}(a^2, b^2, c^2))\\ &= S_{(2m, 2n+1)}(a^2, b^2, c^2),\end{aligned}$$

finishing the proof of Theorem 4.2 by replacing a^2 with a, b^2 with b, and c^2 with c.

Let $f_n^*(x)$ be a modified Fibonacci polynomial, with

$$\begin{cases} f_n^*(x) = f_n(x), \; n \text{ odd},\\ f_n^*(x) = \dfrac{f_n(x)}{x}, \; n \text{ even}.\end{cases}$$

Listing the first few values,

$$\begin{aligned}f_1^*(x) &= 1\\ f_2^*(x) &= 1\\ f_3^*(x) &= x^2 + 1\\ f_4^*(x) &= x^2 + 2\\ f_5^*(x) &= x^4 + 3x^2 + 1\\ f_6^*(x) &= x^4 + 4x^2 + 3\\ f_7^*(x) &= x^6 + 5x^4 + 6x^2 + 1\\ f_8^*(x) &= x^6 + 6x^4 + 10x^2 + 4.\end{aligned}$$

Here,

$$\begin{cases} f_{n+2}^*(x) = f_{n+1}^*(x) + f_n^*(x), \; n \text{ even},\\ f_{n+2}^*(x) = x^2 f_{n+1}^*(x) + f_n^*(x), \; n \text{ odd}.\end{cases}$$

This is $\{S_k(a, b, c, d)\}$ with $a = b = d = 1$, $c = x^2$. Thus, by Theorem 4.2,

$$(f_m^*(x), f_n^*(x)) = f_{(m, n)}^*(x).$$

Let $v_k(x)$ be a modified Morgan-Voyce polynomial defined by

$$v_{2n+2}(x) = B_n(x), \; v_{2n+1}(x) = b_n(x).$$

The first few values for $\{v_k(x)\}$ are

$$\begin{aligned}v_1(x) &= 1 &&= b_0(x)\\ v_2(x) &= 1 &&= B_0(x)\\ v_3(x) &= x + 1 &&= b_1(x)\\ v_4(x) &= x + 3 &&= B_1(x)\\ v_5(x) &= x^2 + 3x + 1 &&= b_2(x)\\ v_6(x) &= x^2 + 4x + 3 &&= B_2(x)\\ v_7(x) &= x^3 + 5x^2 + 6x + 1 &&= b_3(x)\\ v_8(x) &= x^3 + 6x^2 + 10x + 4 = (x+2)(x^2+4x+2) &&= B_3(x)\end{aligned}$$

Since $v_k(x)$ satisfies

$$\begin{cases} v_n(x) = v_{n-1}(x) + v_{n-2}(x), & n \text{ even}, \\ v_n(x) = xv_{n-1}(x) + v_{n-2}(x), & n \text{ odd}, \end{cases}$$

this is $\{S_k(a, b, c, d)\}$ with $a = b = d = 1$ and $c = x$. Then, by Theorem 4.2,

$$(v_n(x), v_m(x)) = v_{(m,n)}(x).$$

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THE GOLDEN SECTION IN THE EARLIEST NOTATED WESTERN MUSIC

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The persistent use of the golden section as a proportion in Western Art is well recognized. Architecture, the visual arts, sculpture, drama, and poetry provide examples of its use from ancient Greece to the present day. No similar persistence has been established in music. One possible reason is that what ancient Greek music has survived is of such a fragmentary nature that it is not possible to make reliable musical deductions from it. However, beginning with the early Middle Ages a large body of music has survived in manuscripts that from ca. 10th century can be read and the music can be performed. This body of music is known as Roman liturgical chant or, more commonly, as Gregorian chant. These chants have not previously been analyzed from the standpoint of the golden section. Acknowledging the probability of the pres-